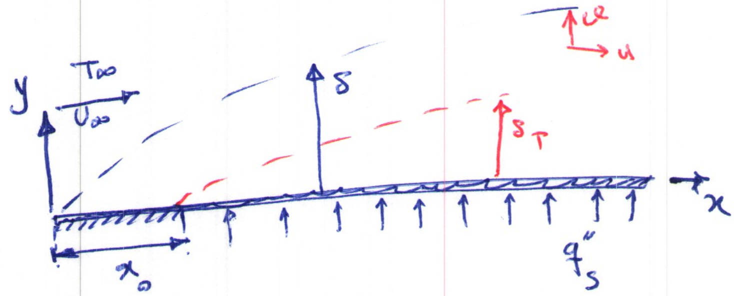
Uniform surface heat flux

Figure shows a flat plate with an insulated leading section of length x_0. The plate is heated with uniform flux q''_s along its surface $x \geq x_0$. Consider steady state, laminar, two-dimensional flow with constant property. Please find the temperature distribution & Nusselt number.



حل انتگرالی :

Equations:

1- Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

2- Momentum $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dP_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$

3- Energy $u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$

که فرم انتگرالی به صورت زیر خواهد بود.

$$\nu\frac{\partial u}{\partial y}\Big|_{y=0} = U_\infty\frac{d}{dx}\int_0^{\delta(x)} u\,dy - \frac{d}{dx}\int_0^{\delta(x)} u^2\,dy$$

$$\alpha\frac{\partial T}{\partial y}\Big|_{y=0} = \frac{d}{dx}\int_0^{\delta_T(x)} u(T_\infty - T)\,dy$$

که پروفیل دما و سرعت را از مرتبه ۳ در نظر بگیریم :

$$u(y) = a_0 + a_1 y + a_2 y^2 + a_3 y^3$$

$y=0 \quad u=0 \longrightarrow a_0 = 0$

$y=\delta \quad u=U_\infty \longrightarrow U_\infty = a_1\delta + a_2\delta^2 + a_3\delta^3 \longrightarrow \boxed{U_\infty = a_1\delta + a_3\delta^3}$

$y=\delta \quad \frac{\partial u}{\partial y}=0 \to a_1 + 2a_2 y + 3a_3 y^2\Big|_{y=\delta} = 0 \longrightarrow a_1 + 2a_2\delta + 3a_3\delta^2 = 0 \Rightarrow \boxed{a_1 + 3a_3\delta^2 = 0}$

$y=0 \quad \frac{\partial^2 u}{\partial y^2}=0 \to 2a_2 + 6a_3 y\Big|_{y=0} = 0 \to a_2 = 0$

$$U_\infty = a_1\delta + a_3\delta^3 \longrightarrow U_\infty = -3a_3\delta^3 + a_3\delta^3 = -2a_3\delta^3$$

$$a_1 + 3a_3\delta^2 = 0 \rightarrow a_1 = -3a_3\delta^2$$

$$\boxed{a_3 = \frac{U_\infty}{-2\delta^3}}$$

$$a_1 = \frac{-3U_\infty}{-2\delta} = \frac{3}{2}\frac{U_\infty}{\delta}$$

$$\boxed{\frac{u}{U_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3}$$

که با جایگذاری این روابط در معادله انتگرال مومنتم.

$$C_f = \frac{0.646}{Re_x^{1/2}} \quad , \quad \frac{\delta(x)}{x} = \frac{4.64}{Re_x^{1/2}}$$

برای معادله انرژی با تخمین درجه ۳ برای دما

$$T(y) = b_0 + b_1 y + b_2 y^2 + b_3 y^3$$

1. $-k\left.\frac{\partial T}{\partial y}\right|_{y=0} = q''_s \rightarrow -k(b_1 + 2b_2 y + 3b_3 y^2)\Big|_{y=0} = q''_s \Rightarrow \left(b_1 = \frac{-q''_s}{k}\right)$

2. $T(x, \delta_T) = T_\infty \longrightarrow T_\infty = b_0 + b_1\delta_T + b_2\delta_T^2 + b_3\delta_T^3$

3. $\frac{\partial T}{\partial y}(x, \delta_T) = 0 \longrightarrow b_1 + 2b_2 y + 3b_3 y^2\Big|_{y=\delta_T} = 0 \Rightarrow b_1 + 2b_2\delta_T + 3b_3\delta_T^2 = 0$

4. $\frac{\partial^2 T}{\partial y^2}(x, 0) = 0 \longrightarrow 2b_2 + 6b_3 y\Big|_{y=0} = 0 \longrightarrow \boxed{b_2 = 0}$

$$-\frac{q''_s}{k} + 3b_3\delta_T^2 = 0 \longrightarrow \boxed{b_3 = \frac{q''_s}{3k\delta_T^2}}$$

$$T_\infty = b_0 + \left(-\frac{q''_s}{k}\right)\delta_T + \frac{q''_s}{3k\delta_T^2}\delta_T^3 \rightarrow b_0 = T_\infty + \frac{q''_s\delta_T}{k} - \frac{q''_s\delta_T}{3k}$$

$$= T_\infty + \frac{2}{3}\frac{q''_s\delta_T}{k}$$

$$T(x,y) = T_\infty + \left[\frac{2}{3}\delta_t - y + \frac{1}{3}\frac{y^3}{\delta_t^2}\right]\frac{q''_s}{k}$$

$$T_s(x) = T(x,0) = T_\infty + \frac{2}{3}\frac{q''_s}{k}\delta_t \longrightarrow T_s(x) - T_\infty = \frac{2}{3}\frac{q''_s}{k}\delta_t$$

$$h(x) = \frac{q''_s}{T_s(x) - T_\infty} \Rightarrow Nu_x = \frac{hx}{k} = \frac{q''_s x}{k[T_s(x) - T_\infty]}$$

$$= \frac{q''_s x}{k\left(\frac{2}{3}\frac{q''_s}{k}\delta_T\right)} = \frac{3}{2}\frac{x}{\delta_T(x)}$$

بنابراین برای محاسبه Nu_x باید $\delta_T(x)$ را تعیین کنیم.

حال $T(x,y)$ ، $u(x,y)$ را در معادله قرار می‌دهیم

$$\alpha \frac{\partial T}{\partial y}\Big|_{y=0} = \frac{d}{dx}\int^{\delta_T(x)} u(T_\infty - T)\,dy$$

$$\cancel{\alpha q''_s} = \alpha\left(-1 + \frac{y^2}{\delta_T^2}\right)\frac{q''_s}{k}\Bigg|_{y=0} = -\frac{\alpha q''_s}{k} = \frac{-d}{dx}\int_0^{\delta_T(x)} u_\infty\left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\frac{y^3}{\delta^3}\right]\left[\frac{2}{3}\delta_T - y + \frac{1}{3}\frac{y^3}{\delta_T^2}\right]\frac{q''_s}{k}\,dy$$

$$+\cancel{\frac{\alpha q''_s}{k}}\ \alpha = \frac{u_\infty d}{dx}\int_0^{\delta_T}\left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\right]\left[\frac{2}{3}\delta_T - y + \frac{1}{3}\frac{y^3}{\delta_T^2}\right]dy$$

$$\frac{\alpha}{u_\infty} = \frac{d}{dx}\int_0^{\delta_T}\left[\left(\frac{\delta_T}{\delta}\right)y - \frac{3}{2}\frac{y^2}{\delta} + \frac{1}{2}\frac{y^4}{\delta_T^2\delta} - \frac{1}{3}\frac{y^3\delta_T}{\delta^3} + \frac{1}{2}\frac{y^4}{\delta^3} - \frac{1}{6}\frac{y^6}{\delta^3\delta_T^2}\right]dy$$

ECP 2-3

$$\frac{\alpha}{U_\infty} = \frac{d}{dx}\left[\cancel{\tfrac{3}{4}}\,\tfrac{1}{2}\left(\frac{\delta_T}{\delta}\right)y^2 - \frac{1}{2}\frac{y^3}{\delta} + \frac{1}{10}\frac{y^5}{\delta_T^2\delta} - \frac{1}{12}\frac{y^4\delta_T}{\delta^3} + \frac{1}{10}\frac{y^5}{\delta^3} - \frac{1}{42}\frac{y^7}{\delta^3\delta_T^2}\right]^{\delta_T}$$

$$= \frac{d}{dx}\left[\cancel{\tfrac{3}{4}}\,\tfrac{1}{2}\delta_T^2\left(\frac{\delta_T}{\delta}\right) - \frac{1}{2}\left(\frac{\delta_T}{\delta}\right)\delta_T^2 + \frac{1}{10}\left(\frac{\delta_T}{\delta}\right)\delta_T^2 - \frac{1}{12}\cancel{\left(\frac{\delta_T}{\delta}\right)^3}\delta_T^2 + \frac{1}{10}\cancel{\left(\frac{\delta_T}{\delta}\right)^3}\delta_T^2 + \frac{1}{42}\cancel{\left(\frac{\delta_T}{\delta}\right)^3}\delta_T^2\right]$$

$$Pr \gg 1 \Leftrightarrow \frac{\delta_T}{\delta} \ll 1 \rightarrow \left(\frac{\delta_T}{\delta}\right)^3 \ll \left(\frac{\delta_T}{\delta}\right)$$

$$\frac{\alpha}{U_\infty} = \frac{d}{dx}\left[\left(\cancel{\tfrac{3}{4}}\,\tfrac{1}{2} - \frac{1}{2} + \frac{1}{10}\right)\left(\frac{\delta_T}{\delta}\right)\delta_T^2\right] = \frac{d}{dx}\left(\frac{1}{10}\delta_T^2\left(\frac{\delta_T}{\delta}\right)\right)$$

$$\cancel{\frac{15-10+2}{20}} = \left(\tfrac{7}{20}\right)$$

$$\frac{10\alpha}{U_\infty} = \frac{d}{dx}\left(\frac{\delta_T^3}{\delta}\right) \rightarrow \frac{\delta_T^3}{\delta} = \frac{10\alpha}{U_\infty}x + C$$

$$\begin{matrix} x = x_0 \\ \delta_T = 0 \end{matrix} \longrightarrow C = \frac{-10\alpha x_0}{U_\infty}$$

$$\frac{\delta_T^3}{\delta} = \frac{10\alpha}{U_\infty}x - \frac{10\alpha}{U_\infty}x_0 \Rightarrow \delta_T = \left(\frac{10\alpha}{U_\infty}(x-x_0)\delta\right)^{1/3}$$

$$\delta_T = \left[\frac{10\alpha}{U_\infty}(x-x_0)\frac{4.64}{Re_x^{1/2}}x\right]^{1/3}$$

GXP2-4

$$\delta_T = \left[\frac{46.4\,\alpha\, x\, \cancel{x\,4.62}}{U_\infty\, x}\,\frac{(x-x_0)}{R_{ex}^{1/2}}\right]^{1/3} = \left[\frac{46.4}{P_{rx}\, R_{ex}^{1/2}}\, x^3 \left(1-\frac{x_0}{x}\right)\right]^{1/3}$$

$$= \frac{3.594\, x}{R_e^{1/2}\, Pr^{1/3}}\left[1-\frac{x_0}{x}\right]^{1/3}$$

$$\frac{\delta_T}{x} = \frac{3.594}{R_{ex}^{1/2}\, Pr^{1/3}}\left[1-\frac{x_0}{x}\right]^{1/3}$$

$$\rightarrow T_s(x) = T_\infty + \frac{2}{3}\,\frac{q_s''}{k}\,\delta_T$$

$$= T_\infty + \frac{2}{3}\,\frac{q_s''}{k} \times \frac{3.594}{R_{ex}^{1/2}\, Pr^{1/3}}\left[1-\frac{x_0}{x}\right]^{1/3}$$

$$= T_\infty + 2.396\,\frac{q_s''}{k}\left[1-\frac{x_0}{x}\right]^{1/3}\frac{x}{Pr^{1/3}\, R_{ex}^{1/2}}$$

$$Nu_x = \frac{3}{2}\,\frac{x}{\delta_T} = \frac{3}{2}\,\frac{1}{\frac{3.594}{Pr^{1/3}\, R_{ex}^{1/2}}\left[1-\frac{x_0}{x}\right]^{1/3}}$$

$$= 0.417\, Pr^{1/3}\, R_{ex}^{1/2}\left[1-\frac{x_0}{x}\right]^{-1/3}$$

برای حالت خاص: $x_0 = 0$ یعنی شار حرارتی از ابتدای صفحه باشد

$$Nu_x = 0.417\, Pr^{1/3}\, R_{ex}^{1/2}$$

$$T_{s(x)} = T_\infty + 2.3\cancel{9}6\,\frac{q_s''}{k}\,\cancel{\left[1-\frac{x_0}{x}\right]}\,\frac{x}{Pr^{1/3}\, R_{ex}^{1/2}}$$

$$= T_\infty + 2.396\,\frac{q_s''}{k}\,\frac{1}{Pr^{1/3}}\left(\frac{x}{\left(\frac{U_\infty x}{\nu}\right)^{1/2}}\right) = T_\infty + 2.396\,\frac{q_s''}{k}\,\frac{1}{Pr^{1/3}}\left(\frac{\nu}{U_\infty}\right)^{1/2}\sqrt{x}$$

$= T_\infty + C\sqrt{x}$ [illegible]

$$\delta_T = \left[\frac{46.4\,\alpha\, x \times x\cancel{4.62}}{U_\infty x}\,\frac{(x-x_0)}{R_{ex}^{1/2}}\right]^{1/3} = \left[\frac{46.4}{P_{rx} R_{ex}^{1/2}}\, x^3\left(1-\frac{x_0}{x}\right)\right]^{1/3}$$

$$= \frac{3.594\, x}{R_e^{1/2} Pr^{1/3}}\left[1-\frac{x_0}{x}\right]^{1/3}$$

$$\frac{\delta_T}{x} = \frac{3.594}{R_{ex}^{1/2} Pr^{1/3}}\left[1-\frac{x_0}{x}\right]^{1/3}$$

$$\rightarrow T_s(x) = T_\infty + \frac{2}{3}\,\frac{q_s''}{k}\,\delta_T$$

$$= T_\infty + \frac{2}{3}\,\frac{q_s''}{k} \times \frac{3.594}{R_{ex}^{1/2} Pr^{1/3}}\left[1-\frac{x_0}{x}\right]^{1/3}$$

$$= T_\infty + 2.396\,\frac{q_s''}{k}\left[1-\frac{x_0}{x}\right]^{1/3}\frac{x}{Pr^{1/3} R_{ex}^{1/2}}$$

$$Nu_x = \frac{3}{2}\,\frac{x}{\delta_T} = \frac{3}{2}\,\frac{1}{\dfrac{3.594}{Pr^{1/3} R_{ex}^{1/2}}\left[1-\dfrac{x_0}{x}\right]^{1/3}}$$

$$= 0.417\, Pr^{1/3} R_{ex}^{1/2}\left[1-\frac{x_0}{x}\right]^{-1/3}$$

برای حالت خاص: $x_0 = 0$ یعنی شار حرارتی از ابتدای صفحه باشه

$$Nu_x = 0.417\, Pr^{1/3} R_{ex}^{1/2}$$

$$T_s(x) = T_\infty + 2.396\,\frac{q_s''}{k}\cancel{\left[1-\frac{x_0}{x}\right]}\,\frac{x}{Pr^{1/3} R_{ex}^{1/2}}$$

$$= T_\infty + 2.396\,\frac{q_s''}{k}\,\frac{1}{Pr^{1/3}}\left(\frac{x}{\left(\frac{U_\infty x}{\nu}\right)^{1/2}}\right) = T_\infty + 2.396\,\frac{q_s''}{k}\,\frac{1}{Pr^{1/3}}\left(\frac{\nu}{U_\infty}\right)^{1/2}\sqrt{x}$$

$$= T_\infty + C\sqrt{x}$$ [illegible]

Exp 2-5