$$\begin{cases} u_t - 4u_{xx} = e^{-ax} \\ u(0,t) = 0 \\ u(l,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

حل مسئله با استفاده از روش بسط برحسب توابع ویژه:

* ابتدا مسئله همگن: شرایط مرزی را حل کرده، توابع ویژه را می‌یابیم:

با جداسازی متغیرها شروع می‌کنیم:

$$\begin{cases} u_t - 4u_{xx} = 0 \\ u(0,t) = 0 \\ u(l,t) = 0 \end{cases} \longrightarrow u(x,t) = F(x)G(t) \longrightarrow F\dot{G} - 4F''G = 0 \longrightarrow \frac{1}{4}\frac{\dot{G}}{G} = \frac{F''}{F} = \begin{cases} k^2 \\ 0 \\ -k^2 \end{cases}$$

$$-k^2 \longrightarrow \frac{F''}{F} = -k^2 \longrightarrow F'' + k^2 F = 0 \quad F(x) = A\cos kx + B\sin kx$$

$$u(0,t) = 0 \longrightarrow F(0) = 0 \longrightarrow A = 0$$

$$u(l,t) = 0 \longrightarrow F(l) = 0 \longrightarrow B\sin kl = 0 \xrightarrow{B \neq 0} \sin kl = 0, \quad kl = n\pi, \quad k = \frac{n\pi}{l}$$

$$F_n(x) = B_n \sin\frac{n\pi}{l}x \longrightarrow \quad \sin\frac{n\pi}{l}x = \text{تابع ویژه}$$

$$u(x,t) = \sum_{n=1}^{\infty} G_n(t)\sin\frac{n\pi}{l}x \longrightarrow$$ فرض می‌کنیم پاسخ مسئله اصلی به صورت روبرو باشد

حال در معادله اصلی قرار می‌دهیم:

$$u_t = \frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \dot{G}_n(t)\sin\frac{n\pi}{l}x$$

$$u_x = \frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} G_n(t)\left(\frac{n\pi}{l}\right)\cos\frac{n\pi}{l}x \longrightarrow u_{xx} = \sum_{n=1}^{\infty} -G_n(t)\left(\frac{n\pi}{l}\right)^2 \sin\frac{n\pi}{l}x$$

در معادله قرار می‌دهیم:

$$\sum_{n=1}^{\infty} \dot{G}_n(t)\sin\frac{n\pi}{l}x + \sum_{n=1}^{\infty}\left(\frac{2n\pi}{l}\right)^2 G_n(t)\sin\frac{n\pi}{l}x = e^{-ax}$$

$$\sum_{n=1}^{\infty}\left(\dot{G}_n(t) + \left(\frac{2n\pi}{l}\right)^2 G_n(t)\right)\sin\frac{n\pi}{l}x = e^{-ax}$$

$$\dot{G}_n(t) + \left(\frac{2n\pi}{l}\right)^2 G_n(t) = \frac{2}{l}\int_0^l e^{-ax}\sin\frac{n\pi}{l}x\,dx = h_n(x)$$

$$\dot{G}_n(t) + \left(\frac{2n\pi}{l}\right) G_n(t) = h_n(x)$$

$$G_n(t) = G_{n,h}(t) + G_{n,p}(t)$$

$$\dot{G}_{n,h}(t) + \left(\frac{2n\pi}{l}\right)^2 G_n(t) = 0 \Rightarrow \frac{dG_n}{G_n} = -\left(\frac{2n\pi}{l}\right)^2 dt \longrightarrow G_{n,h}(t) = c^* e^{-\left(\frac{2n\pi}{l}\right)^2 t}$$

$$G_n(t) = A^* t + B^* \longrightarrow A^* + \left(\frac{2n\pi}{l}\right)A^* t + \left(\frac{2n\pi}{l}\right)B^* = h_n(x)$$

$$G_n(t) = c^* e^{-\left(\frac{2n\pi}{l}\right)^2 t} + \left(\frac{l}{2n\pi}\right)^2 h_n(x)$$

$$h_n(x) = \frac{2}{l}\int_0^l e^{-ax}\sin\frac{n\pi}{l}x\,dx = \frac{2}{l}\left[\frac{-l}{n\pi}e^{-ax}\cos\frac{n\pi}{l}x\Big|_0^l - \int_0^l \frac{+l}{n\pi}\cos\frac{n\pi}{l}x\,(a\,e^{-ax})\,dx\right]$$

$$e^{-ax} = u \rightarrow du = -a e^{-ax}dx$$

$$\sin\frac{n\pi}{l}x\,dx = dv \rightarrow v = \frac{-l}{n\pi}\cos\frac{n\pi}{l}x$$

$$\frac{al}{n\pi}\int_0^l e^{-ax}\cos\frac{n\pi}{l}x\,dx = \frac{al}{n\pi}\left[\cancel{e^{-ax}\frac{l}{n\pi}\sin\frac{n\pi}{l}x\Big|_0^l} + \int_0^l \frac{la}{n\pi}e^{-ax}\sin\frac{n\pi}{l}x\,dx\right]$$

$$e^{-ax} = u \rightarrow -ae^{-ax}dx = du$$

$$\cos\frac{n\pi}{l}x\,dx = dv \rightarrow v = \frac{l}{n\pi}\sin\frac{n\pi}{l}x$$

$$h_n(x) = \frac{2}{l}\int_0^l e^{-ax}\sin\frac{n\pi}{l}x\,dx = \frac{2}{l}\left[\frac{-l}{n\pi}\left[e^{-al}\cos n\pi - 1\right] + \frac{a^2l^2}{n^2\pi^2}\int_0^l e^{-ax}\sin\frac{n\pi}{l}x\,dx\right]$$

$$= \frac{-2}{n\pi}\left[e^{-al} - 1\right] + \frac{2}{l}\left(\frac{a^2l^2}{n^2\pi^2}\right)\int_0^l e^{-ax}\sin\frac{n\pi}{l}x\,dx$$

$$\left[1 - \frac{a^2l^2}{n^2\pi^2}\right]\frac{2}{l}\int_0^l e^{-ax}\sin\frac{n\pi}{l}x\,dx = \frac{+2}{n\pi}\left[1 - e^{-al}\right]$$

$$h_n(x) = \frac{2}{l}\int_0^l e^{-ax}\sin\frac{n\pi}{l}x\,dx = \frac{\frac{2}{n\pi}\left[1 - e^{-al}\right]}{n^2\pi^2 - a^2l^2}\times(n\pi)^2 = \frac{2n\pi\left[1 - e^{-al}\right]}{n^2\pi^2 - a^2l^2}$$

$$C_n(t) = c^* e^{-\left(\frac{2n\pi}{l}\right)^2 t} + \frac{l^2}{2n\pi}\left[\frac{1 - e^{-al}}{n^2\pi^2 - a^2l^2}\right]$$

$$u(x,t) = \sum_{n=1}^{\infty}\left(c^* e^{-\left(\frac{2n\pi}{l}\right)^2 t} + \frac{l^2}{2n\pi}\left[\frac{1 - e^{-al}}{n^2\pi^2 - a^2l^2}\right]\right)\sin\frac{n\pi}{l}x$$