ریاضی مهندسی

دانشگاه سمنان - دانشکده مکانیک

Fall 2016

$$\begin{cases} u_t = \alpha^2 u_{xx} + \sin(3\pi x) & t>0,\ 0<x<1 \\ u(x,0) = \sin(\pi x) & 0<x<1 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

① حل مسأله همگن، بدست آوردن توابع ویژه

$$\begin{cases} u_t = \alpha^2 u_{xx} \\ u(0,t) = u(1,t) = 0 \end{cases} \longrightarrow u(x,t) = F(x)\,G(t) \longrightarrow \alpha^2 F'' G = F \dot{G} \xrightarrow{\div FG} \frac{1}{\alpha^2}\frac{\dot{G}}{G} = \frac{F''}{F}$$

$$\frac{1}{\alpha^2}\frac{\dot{G}}{G} = \frac{F''}{F} = \begin{cases} k^2 & \times \\ 0 & \times \\ -k^2 \end{cases} \Longrightarrow F'' + k^2 F = 0 \longrightarrow F(x) = A\cos kx + B\sin kx$$

$$\Longrightarrow F(0) = 0 \longrightarrow A = 0$$

$$F(1) = 0 \longrightarrow B\sin k = 0 \xrightarrow{B \neq 0} \sin k = 0 \longrightarrow \boxed{k = n\pi}$$

$$F_n(x) = B_n \sin(n\pi x)$$

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بنابراین توابع مشخصه برابرند $F_n(x) = \sin(n\pi x)$

$$\rightarrow u(x,t) = \sum_{n=1}^{\infty} G_n(t)\sin(n\pi x)$$

$$u_t(x,t) = \sum_{n=1}^{\infty} \dot{G}_n(t)\sin n\pi x$$

$$\frac{\partial u}{\partial x} = u_x(x,t) = \sum_{n=1}^{\infty} G_n(t)(n\pi)\cos(n\pi x)$$

$$\frac{\partial^2 u}{\partial x^2} = u_{xx}(x,t) = \sum_{n=1}^{\infty} G_n(t)(-n^2\pi^2)\sin(n\pi x)$$

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$$\sum_{n=1}^{\infty} \dot{G}_n(t)\sin(n\pi x) = \alpha^2 \sum_{n=1}^{\infty} G_n(t)(-n^2\pi^2)\sin(n\pi x) + \sin(3\pi x)$$

$$\sum_{n=1}^{\infty}\left[\dot{G}_n(t) + \alpha^2 n^2 \pi^2 G_n(t)\right]\sin(n\pi x) = \sin 3\pi x$$

$$\dot{G}_n(t) + \alpha^2 n^2 \pi^2 G_n(t) = 2\int_0^1 \sin(3\pi x)\sin(n\pi x)\,dx = \begin{cases} 0 & n \neq 3 \\ 1 \end{cases}$$

$$\int_0^1 \sin(3\pi x)\sin n\pi x\,dx = \frac{1}{2}\int_0^1 [\cos(3\pi + n\pi)x - \cos(3\pi - n\pi)x]\,dx = 0 \qquad n \neq 3$$

$$\int_0^1 \sin^2(3\pi x)\,dx = \int_0^1 \frac{1-\cos(6\pi x)}{2}\,dx = \int_0^1 \left[\frac{1}{2} - \frac{1}{2}\cos(6\pi x)\right]dx$$

$$= \frac{1}{2}x\Big|_0^1 - \frac{1}{12\pi}\sin(6\pi x)\Big|_0^1 = \boxed{\frac{1}{2}}$$

+1

$$\dot{G}_n(t) + \alpha^2 n^2 \pi^2 G_n(t) = 1 \longrightarrow$$

$$\dot{G}_n(t) + \lambda_n^2 G_n(t) = 1 \quad , \quad \boxed{\lambda_n = \alpha n \pi}$$

$$\dot{G}_n(t) + \lambda_n^2 G_n = 0 \longrightarrow G_{n_h}(t) = Ce^{-\lambda_n^2 t} \longrightarrow$$

$$G_{n,h}(t) = Ce^{-\lambda_n^2 t}$$

$$G_{n,c}(t) = A \longrightarrow 0 + \lambda_n^2 A = 1 \longrightarrow A = \frac{1}{\lambda_n^2}$$

$$\longrightarrow \boxed{G_n(t) = C_n e^{-\lambda_n^2 t} + \frac{1}{\lambda_n^2}}$$

+1

$$\longrightarrow u(x,t) = \sum_{n=1}^{\infty}\left(C_n e^{-\lambda_n^2 t} + \frac{1}{\lambda_n^2}\right)\sin(n\pi x)$$

$$u(x,0) = \sin\pi x = \sum_{n=1}^{\infty}\left(C_n + \frac{1}{\lambda_n^2}\right)\sin(n\pi x)$$

$$\longrightarrow \left(C_n + \frac{1}{\lambda_n^2}\right) = 2\int_0^1 \sin\pi x \sin n\pi x\,dx = \begin{cases} 0 & n \neq 1 \\ 1 & n = 1 \end{cases}$$

$$C_1 + \frac{1}{\lambda_1^2} = 1 \longrightarrow \boxed{C_1 = 1 - \frac{1}{\lambda_1^2}} \quad , \quad C_n = -\frac{1}{\lambda_n^2} \quad \boxed{n \geq 2}$$

$$u(x,t) = \left(1 - \frac{1}{\alpha^2\pi^2}\right)\sin \pi x - \sum_{n=2} \frac{1}{\alpha^2 n^2 \pi^2} e^{-(\alpha^2 n^2 \pi^2)t} \sin(n\pi x)$$ ٥+٣

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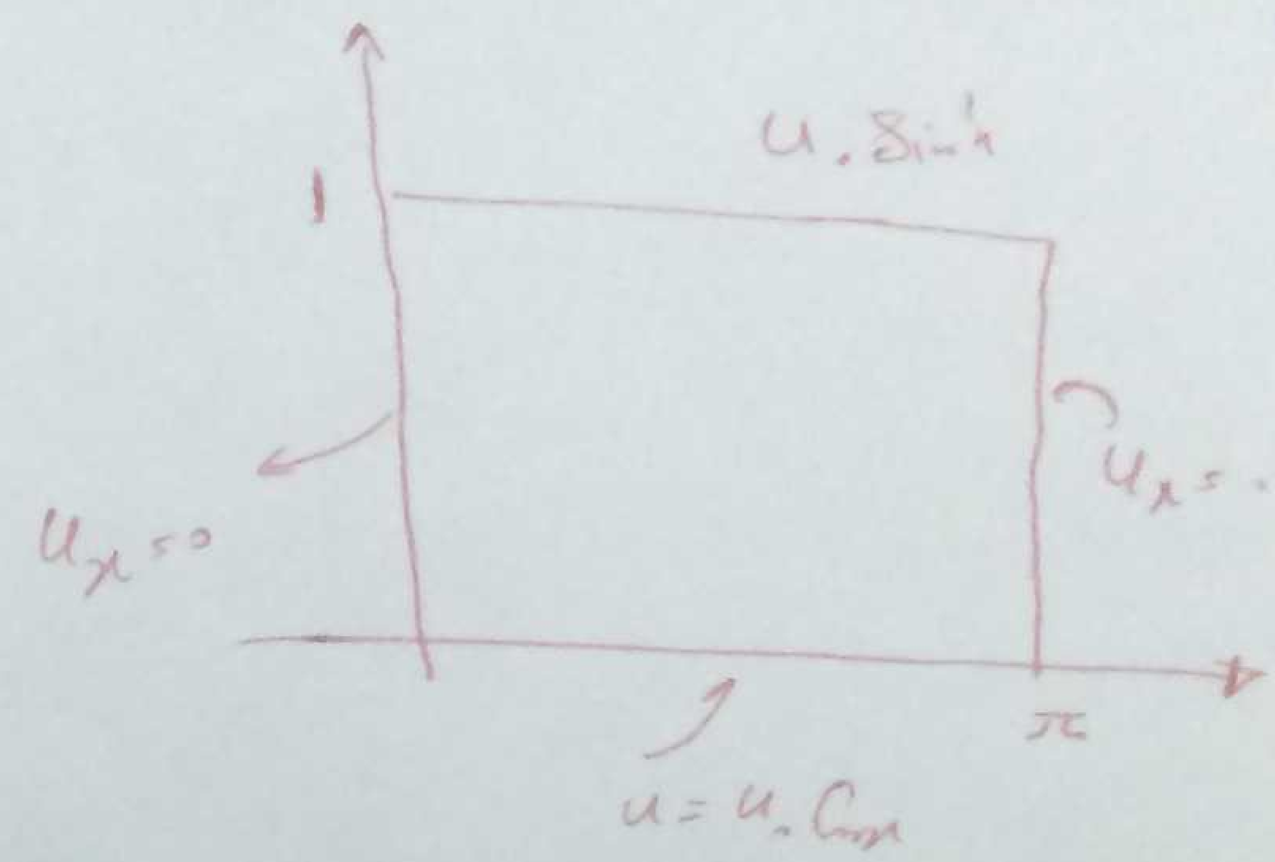
$$\begin{cases} U_{xx} + U_{yy} = 0 & 0<x<\pi \\ & 0<y<1 \\ U_x(0,y) = U_x(\pi,y) = 0 \\ U(x,0) = U_0 \cos x \\ U(x,1) = U_0 \sin^2 x \end{cases}$$



جداسازی متغیرها

$$u(x,y) = X(x)\,Y(y) \longrightarrow X''Y + Y''X = 0 \xrightarrow{\div XY} \frac{X''}{X} + \frac{Y''}{Y} = 0$$

در این حالت جواب غیر صفر شود

$$\frac{X''}{X} = -\frac{Y''}{Y} = \begin{cases} k^2 \\ 0 \\ -k^2 \end{cases}$$

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0 ***

$$\frac{X''}{X} = 0 \longrightarrow X'' = 0 \longrightarrow X(x) = Ax + B$$

شرایط مرزی

$$u_x(0,y) = X'(0)\,Y(y) = 0 \longrightarrow X'(0) = 0$$

$$u_x(\pi,y) = X'(\pi)\,Y(y) = 0 \longrightarrow X'(\pi) = 0$$

$$\left.\begin{matrix} X'(0) = 0 \longrightarrow A = 0 \\ X'(\pi) = 0 \end{matrix}\right\} \longrightarrow \boxed{X_0(x) = B}$$

پس به ازای $k=0$ ما جواب داریم.

$$-\frac{Y''}{Y} = 0 \longrightarrow Y'' = 0 \longrightarrow \boxed{Y_0(y) = Cy + D}$$

$$u_0(x,y) = X_0(x)\,Y_0(y) = \boxed{A_0 y + B_0}$$

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$$\frac{X''}{X} = k^2 \longrightarrow X'' - k^2 X = 0 \longrightarrow X(x) = Ae^{kx} + Be^{-kx}$$

$+k^2$ ***

$$X'(0) = 0 \longrightarrow Ake^{kx} - Bke^{-kx}\Big|_{x=0} = 0 \longrightarrow (A-B)k = 0 \longrightarrow \boxed{A = B}$$

$$X'(\pi) = 0 \longrightarrow Ake^{k\pi} - Bke^{-k\pi} = 0 \longrightarrow Ae^{k\pi} - Be^{-k\pi} = 0$$

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$A = B = 0$

ص ۵

$$-k^2$$

$$\longrightarrow \frac{X''}{X} = -k^2 \longrightarrow X'' + k^2 X = 0 \longrightarrow X_{(x)} = A\cos kx + B\sin kx$$

$$X'(0) = 0 \longrightarrow X'_{(x)} = -Ak\sin kx + Bk\cos kx\Big|_{x=0} = 0 \rightarrow \boxed{B = 0}$$

$$X'(\pi) = 0 \longrightarrow X'_{(\pi)} = -Ak\sin kx\Big|_{x=\pi} = 0 = -Ak\sin k\pi = 0$$

$$A \neq 0,\ k \neq 0 \longrightarrow \sin k\pi = 0 \longrightarrow k\pi = n\pi \longrightarrow k = n,\quad n = 1, 2, 3, 4 \dots$$

$$\boxed{X_{n(x)} = A_n \cos nx}$$ ۴

$$-\frac{Y''}{Y} = n^2 \longrightarrow Y''_{(y)} - n^2 Y = 0 \longrightarrow Y_{n(y)} = C_n \cosh ny + D_n \sinh ny$$

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$$\longrightarrow u_n(x,y) = A_0 + B_0 y + \sum_{n=1}^{\infty} (C'_n \cosh ny + D'_n \sinh ny)\cos nx$$ ۲

برای تعیین A_0 ، B_0 ، C'_n ، D'_n از شرط مرزی استفاده می‌کنیم.

$$u(x,0) = A_0 + \sum_{n=1}^{\infty} C'_n \cos nx = u_0 \cos x \longrightarrow A_0 = 0$$

$$C'_1 = u_0$$

$$C'_2 = C'_3 = C'_4 \dots = 0$$ ۴

$$\longrightarrow u(x,y) = B_0 y + u_0 \cosh y \cos x + \sum_{n=1}^{\infty} D'_n \sinh ny \cos nx$$

$$\longrightarrow u(x,1) = B_0 + u_0 \cosh 1 \cos x + \sum_{n=1}^{\infty} D'_n \sinh n \cos nx = u_0 \sin^2 x$$

$$u_0\left(\frac{1-\cos 2x}{2}\right) = B_0 + u_0 \cosh 1 \cos x + \sum D'_n \sinh n \cos nx$$

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$$\longrightarrow B_0 = \frac{u_0}{2}, \quad B_1 = \frac{-u_0 \cosh 1}{\sinh 1}, \quad B_2 = -\frac{u_0}{2\sinh 2}, \quad B_n = 0 \text{ for } n \geq 3$$

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$$\longrightarrow u(x,y) = \frac{u_0}{2} y + u_0 \left[\cosh y - \frac{\cosh 1 \sinh y}{\sinh 1}\right] \cos x - u_0 \frac{\sinh 2y}{2\sinh 2} \cos 2x$$

$$= u_0 \left\{ \frac{1}{2} y + \frac{\sinh(1-y)}{\sinh 1} \cos x - \frac{\sinh 2y}{2\sinh 2} \cos 2x \right\}$$

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$$\begin{cases} \left[u_{xx} - u_{tt} = e^{-\pi^2 t} \sin \pi x\right] \\ u(0,t) = 0 \\ u_x(1,t) = 0 \\ u(x,0) = 0 \\ u_t(x,0) = 0 \end{cases}$$

$$U_{xx}(x,s) - \left[s^2 U(x,s) - \cancel{s u(x,0)} - \cancel{u_t(x,0)}\right] = \frac{\sin \pi x}{s+\pi^2}$$

$$\begin{cases} U_{xx}(x,s) - s^2 U(x,s) = \dfrac{\sin \pi x}{s+\pi^2} \\ U(0,s) = 0 \\ U_x(1,s) = 0 \end{cases}$$

$$U_{xx} - s^2 U = 0$$

پاسخ عمومی

$$U_h(x,s) = A e^{-sx} + B e^{sx}$$

پاسخ خصوصی

$$U_p(x,s) = C \sin \pi x + D \cos \pi x$$

$$U_{p,x} = C\pi \cos \pi x - D\pi \sin \pi x$$

$$U_{p,xx} = -C\pi^2 \sin \pi x - D\pi^2 \cos \pi x$$

$$-C\pi^2 \sin \pi x - D\pi^2 \cos \pi x - s^2 C \sin \pi x - s^2 D \cos \pi x = \frac{\sin \pi x}{s+\pi^2}$$

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$$-C(\pi^2+s^2)\sin\pi x - D(\pi^2+s^2)\cos\pi x = \frac{\sin\pi x}{s+\pi^2}$$

$$-C(\pi^2+s^2) = \frac{1}{s+\pi^2} \longrightarrow C = \frac{-1}{(\pi^2+s^2)(s+\pi^2)}$$

$$D(\pi^2+s^2)=0 \longrightarrow D=0$$

$$U_p(x,s) = \frac{-\sin\pi x}{(\pi^2+s^2)(s+\pi^2)}$$

$$U(x,s) = Ae^{-sx} + Be^{sx} - \frac{\sin\pi x}{(s^2+\pi^2)(s+\pi^2)}$$

$$U(0,s) = A+B=0$$

$$U'(1,s) = -sAe^{-sx} + sBe^{sx} - \left.\frac{\pi\cos\pi x}{(s^2+\pi^2)(s+\pi^2)}\right|_{x=1} = 0$$

$$= s(Ae^{s} - Be^{-s}) + \frac{\pi}{(s+\pi^2)(s^2+\pi^2)}\cos\pi = 0$$

$$A = -B = \frac{-\pi}{2s(s+\pi^2)(s^2+\pi^2)\sinh s}$$

حل پایای انتقال حرارت در استوانه‌ی توپر با شعاع a :

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{\partial^2 u}{\partial z^2}=0 \\ \frac{\partial u}{\partial r}(a,z)=0 \quad (1) \\ u(r,0)=f(r) \\ u(r,l)=g(r) \end{cases}$$

حل با استفاده از جداسازی متغیرها :

$$u(r,z)=R(r)Z(z)$$

$\Big\Downarrow$ مشتق‌گیری و جایگذاری در معادله اصلی

$$\frac{1}{r}\frac{\partial}{\partial r}(rR'Z)+RZ''=0$$

$$\Big\Downarrow \div RZ$$

$$\frac{1}{rR}\frac{\partial}{\partial r}(rR')=-\frac{Z''}{Z}=\begin{cases}k^2\\0\\-k^2 \checkmark\end{cases}$$

با توجه به همگن بودن شرط در راستای r ، $-k^2$ را انتخاب می‌کنیم :

$$\therefore \frac{1}{rR}\frac{d}{dr}(rR')=-k^2$$

$$\Rightarrow \frac{1}{rR}(R'+rR'')=-k^2$$

$$\Rightarrow r\frac{d^2R}{dr^2}+\frac{dR}{dr}+k^2rR=0 \quad (\mathrm{I})$$

با انتخاب $x=kr$ داریم :

$$\frac{dR}{dr}=\frac{dR}{dx}\cdot\frac{dx}{dr}=k\frac{dR}{dr}$$

$$\frac{d^2R}{dr^2}=\frac{d}{dr}\left(\frac{dR}{dr}\right)=\frac{d}{dr}\left(k\frac{dR}{dx}\right)$$

$$=\frac{d}{dx}\left(k\frac{dR}{dx}\right)\frac{dx}{dr}=k^2\frac{d^2R}{dx^2}$$

با جایگذاری در (I) داریم :

$$rk^2\frac{d^2R}{dx^2}+k\frac{dR}{dx}+k^2rR=0$$

$$\xrightarrow{\times r} r^2k^2\frac{d^2R}{dx^2}+kr\frac{dR}{dx}+k^2r^2R=0$$

با توجه به اینکه $x=kr$:

$$\Rightarrow x^2\frac{d^2R}{dx^2}+x\frac{dR}{dx}+x^2R=0$$

که معادله‌ی فوق ، معادله‌ی بسل از مرتبه صفر است

$$\Rightarrow R(x)=AJ_0(x)+BY_0(x)$$

$$\xRightarrow{x=kr} R(r)=AJ_0(kr)+BY_0(kr)$$

$$\lim_{r\to 0}|u(r,z)|<\infty \Rightarrow \lim_{r\to 0}R(r)<\infty \quad \Rightarrow B=0$$

$$\Rightarrow R(r)=AJ_0(kr)$$

که مقدار ثابت A با استفاده از شرط مرزی (1) داریم :

$$u_r(a,z)=0 \Longrightarrow R'(a)Z(z)=0$$

$$\xRightarrow{Z(z)\neq 0} R'(a)=0$$

$$\Rightarrow AJ_0'(ka)=0 \xRightarrow{A\neq 0} J_0'(ka)=0$$

$$\Rightarrow ka=\alpha_m \rightarrow$$

می‌دانیم $J_0'(r)=-J_1(r)$ ، در نتیجه k_m ها به صفرهای تابع بسل مرتبه اول وابسته‌اند

$$\Rightarrow \boxed{k_m=\frac{\alpha_m}{a}} \qquad (m=1,2,3,\dots)$$

که در آن $\alpha_1=3.8317, \alpha_2=7.0156, \alpha_3=10.1735$

$$\Rightarrow I_m e^{kl} + E_m e^{-kl} = \beta_m$$

$$e^{-kl} = \frac{1}{p} \Leftarrow e^{kl} = p$$ به فرض

$$\Rightarrow p I_m + \frac{1}{p} E_m = \beta_m \quad (\mathrm{II})$$

$$\xrightarrow[\times p]{(\mathrm{I})} p I_m + p E_m = p\alpha_m$$

$$\xrightarrow{\times(-1)} -pI_m - pE_m = -p\alpha_m \quad (\mathrm{I})'$$

$$\xrightarrow{(\mathrm{I})', (\mathrm{II})} \frac{1-p}{p} E_m = \beta_m - p\alpha_m$$

$$\Rightarrow E_m = \frac{p(\beta_m - p\alpha_m)}{1-p}$$

$$\Rightarrow E_m = \frac{e^{kl}(\beta_m - e^{kl}\alpha_m)}{1 - e^{kl}}$$

و با جایگذاری E_m در (II) داریم:

$$I_m = \frac{\beta_m}{p} - \frac{1}{p^2} E_m$$

$$= \frac{\beta_m}{e^{kl}} - \frac{1}{e^{kl}} \left(\frac{\beta_m - e^{kl}\alpha_m}{1 - e^{kl}} \right)$$

$$\Rightarrow R_m(r) = A_m J_0(k_m r)$$

$$\therefore -\frac{Z''}{Z} = -k^2$$

$$\Rightarrow Z'' - k^2 Z = 0$$

$$\Rightarrow Z(z) = C_m e^{kz} + D_m e^{-kz}$$

$$\Rightarrow u(r,z) = R(r) Z(z)$$

$$= \sum_{m=1}^{\infty} \left(I_m e^{kz} + E_m e^{-kz} \right) J_0(k_m r)$$

حال برای محاسبه I_m , E_m می توان از شرایط مرزی استفاده کرد:

$$\xrightarrow{1:} u(r,0) = f(r)$$

$$\Rightarrow u(r,0) = \sum_{m=1}^{\infty} \underbrace{(I_m + E_m)}_{\alpha_m} J_0(k_m r) = f(r)$$

$$\Rightarrow \sum_{m=1}^{\infty} \alpha_m J_0(k_m r) = f(r)$$

که با توجه به ق. تعامد و تابع وزنی توابع بسل در فاصله $[0,a]$ که برابر با r می باشد داریم:

$$\alpha_m = \frac{\int_0^a r f(r) J_0(k_p r)\, dr}{\int_0^a J_0^2(k_p r)\, r\, dr}$$

$$\Rightarrow I_m + E_m = \alpha_m \quad (\mathrm{I})$$

$$\xrightarrow{2:} u(r,l) = g(r)$$

$$\Rightarrow u(r,l) = \sum_{1}^{\infty} \big(\underbrace{I_m e^{kl} + E_m e^{-kl}}_{\beta_m} \big) J_0(k_m r) = g(r)$$

$$\beta_m = \frac{\int_0^a r g(r) J_0(k_p r)\, dr}{\int_0^a J_0^2(k_p r)\, r\, dr}$$

$$\begin{cases} u_{tt} + u_{xxxx} = 0 \qquad 0 < x < 1 \quad , t > 0 \\ u(0,t) = u(l,t) = 0 \\ u_{xx}(0,t) = u_{xx}(l,t) = 0 \\ u_t(x,0) = \sin(\pi x) \\ u(x,0) = \sin(\pi x) + 0.5 \sin(3\pi x) \end{cases}$$

با توجه به همگن بودن شرایط مرزی از جداسازی متغیرها استفاده می‌شود

$$u(x,t) = F(x)\,G(t) \longrightarrow F^{(4)}G + \ddot{G}F = 0 \longrightarrow \frac{F^{(4)}}{F} = -\frac{\ddot{G}}{G} = \begin{cases} \lambda^4 \\ 0 \\ -\lambda^4 \end{cases}$$

$\therefore \lambda^4$

$$\frac{F^{(4)}}{F} = \lambda^4 \rightarrow F^{(4)} - \lambda^4 F = 0 \longrightarrow m^4 - \lambda^4 = 0 \rightarrow (m^2 - \lambda^2)(m^2 + \lambda^2) = 0$$

$$m^2 - \lambda^2 = 0 \rightarrow m = \pm\lambda$$

$$m^2 + \lambda^2 = 0 \rightarrow m = \pm i\lambda$$

$$\longrightarrow F(x) = A\cosh\lambda x + B\sinh\lambda x + C\cos\lambda x + D\sin\lambda x$$

$\therefore \lambda^4$

$$-\frac{\ddot{G}}{G} = -\lambda^4 \longrightarrow \ddot{G} - \lambda^4 G = 0 \qquad G = Ae^{-\lambda^2 t} + Be^{+\lambda^2 t}$$

$\therefore 0$

$$\frac{F^{(4)}}{F} = 0 \longrightarrow F^{(4)} = 0 \rightarrow F^{3} = C \rightarrow F'' = Cx + D \quad , F' = \tfrac{1}{2}Cx^2 + Dx + E$$

$$F = \frac{1}{2}\cdot\frac{1}{3}Cx^3 + \frac{1}{2}Dx^2 + Ex + f$$

$$u(0,t) = F(0)G(t) = 0 \longrightarrow F(0) = 0 \longrightarrow \boxed{f = 0}$$

$$u(l,t) = F(l)G(t) = 0 \longrightarrow F(l) = 0 \rightarrow \frac{1}{6}Cl^3 + \frac{1}{2}Dl^2 + El = 0$$

$$u''(0,t) = F''(0)G(t) = 0 \longrightarrow F''(0) = 0 \longrightarrow \boxed{D = 0}$$

$$u''(l,t) = F''(l)G(t) = 0 \longrightarrow Cl = 0 \longrightarrow \boxed{C = 0}$$

$$\rightarrow \boxed{E = 0}$$

$\longrightarrow F(x) = A\cosh\lambda x + B\sinh\lambda x + C\cos\lambda x + D\sin\lambda x$

$F(0)=0 \longrightarrow \boxed{A + C = 0}$ (I)

$F(l)=0 \longrightarrow A\cosh\lambda l + B\sinh\lambda l + C\cos\lambda l + D\sin\lambda l = 0$

$F''(0)=0 \longrightarrow F' = A\lambda\sinh\lambda x + B\lambda\cosh\lambda x - C\lambda\sin\lambda x + D\lambda\cos\lambda x$

$F'' = A\lambda^2\cosh(\lambda x) + B\lambda^2\sinh(\lambda x) - C\lambda^2\cos\lambda x - D\lambda^2\sin\lambda x$

$F''(0)=0 \longrightarrow A\lambda^2 - C\lambda^2 = 0 \longrightarrow \boxed{A = C}$ (II)

I و II $\longrightarrow \boxed{A = C = 0}$

$F''(l) = B\lambda^2\sinh(\lambda l) - D\lambda^2\sin(\lambda l) = 0$ (III)

$F(l) = 0$ $(B\sinh(\lambda l) + D\sin(\lambda l) = 0)\lambda^2$ (IV)

$\xrightarrow{\text{III} - \text{IV}} 2D\lambda^2\sin\lambda l = 0 \longrightarrow \boxed{D\sin\lambda l = 0}$

$\xrightarrow{\text{III} + \text{IV}} 2B\lambda^2\sinh(\lambda l) = 0 \longrightarrow \boxed{B\sinh(\lambda l) = 0}$

$\sinh(\lambda l) \neq 0$ بنابراین $B = 0$ است

$D\sin\lambda l = 0 \xrightarrow{D \neq 0} \sin\lambda l = 0 \longrightarrow \lambda l = n\pi \longrightarrow \lambda = \frac{n\pi}{l}$

طبق مسئله $l = 1$

$\boxed{\lambda_n = n\pi}$

$n = 1, 2, 3$

$$F_n(x) = D_n \sin \frac{n\pi}{l} x = D_n \sin(n\pi x)$$

$$\rightarrow -\frac{\ddot{G}}{G} = \lambda^4 \rightarrow \ddot{G} + \lambda^4 G = 0 \rightarrow G(t) = E \cos \lambda^2 t + L \sin \lambda^2 t$$

$$G_n(t) = E_n \cos(n\pi)^2 t + L_n \sin(n\pi)^2 t$$

$$\rightarrow u_n(x,t) = \left[a_n \cos(n\pi)^2 t + b_n \sin(n\pi)^2 t\right] \sin n\pi x$$

$$u(x,t) = \sum_{n=1}^{\infty} \left[a_n \cos(n\pi)^2 t + b_n \sin(n\pi)^2 t\right] \sin n\pi x$$

برای محاسبه a_n ، b_n از شرایط اولیه استفاده می‌کنیم

$$u(x,0) = \sin(\pi x) + 0.5 \sin(3\pi x) = \sum_{n=1}^{\infty} a_n \sin n\pi x$$

$$a_n = 2\int_0^1 \left[\sin \pi x + 0.5 \sin(3\pi x)\right] \sin n\pi x \, dx$$ اگر تابع را در بازه [0,1] گسترش فرد دهیم:

$$= 2\int_0^1 \sin \pi x \sin n\pi x \, dx + \int_0^1 \sin(3\pi x) \sin n\pi x \, dx$$

$$a_1 = 2\int_0^1 \sin^2 \pi x \, dx + \cancel{\int_0^1 \sin(3\pi x) \sin \pi x \, dx}^{\,0}$$

$$= 2\int_0^1 \frac{1 - \cos 2\pi x}{2} dx = \int_0^1 dx - \int_0^1 \cos 2\pi x \, dx$$

$$= 1 - \frac{1}{2\pi} \sin 2\pi x \Big|_0^1 = \boxed{1}$$

$$a_3 = 2\int_0^1 \cancel{\sin \pi x \sin 3\pi x \, dx} + \int_0^1 \sin^2 3\pi x \, dx$$

$$= \int_0^1 \frac{1 - \cos 3\pi x}{2} dx = \boxed{\frac{1}{2}}$$

$$a_2 = 0, \quad a_n = 0 \quad , n > 3$$

$$u_t(x,0) = \sum_{n=1}^{\infty}\left[a_n(n\pi)^2 \sin(n\pi)^2 t + b_n(n\pi)^2 \cos(n\pi)^2 t\right]\sin n\pi x \Big|_{t=0} = \sin(\pi x)$$

$$= \sum_{n=1}^{\infty} b_n (n\pi)^2 \sin n\pi x = \sin(\pi x)$$

$$b_1 = \frac{1}{(\pi)^2}, \quad b_n = 0 \quad n>1$$

$$\longrightarrow u(x,t) = \left[\cos\left((\pi)^2 t\right) + \frac{1}{2}\cos\left((3\pi)^2 t\right) + \frac{1}{\pi^2}\sin(\pi^2 t)\right]\sin \pi x$$

$$+ \left[\frac{1}{2}\cos\left((3\pi)^2 t\right)\right]\sin(3\pi x)$$

$$\boxed{u(x,t) = \left[\cos \pi^2 t + \frac{1}{\pi^2}\sin \pi^2 t\right]\sin(\pi x) + \frac{1}{2}\cos 9\pi^2 t \sin(3\pi x)}$$

ص ۱۴ ریاضی مهندسی

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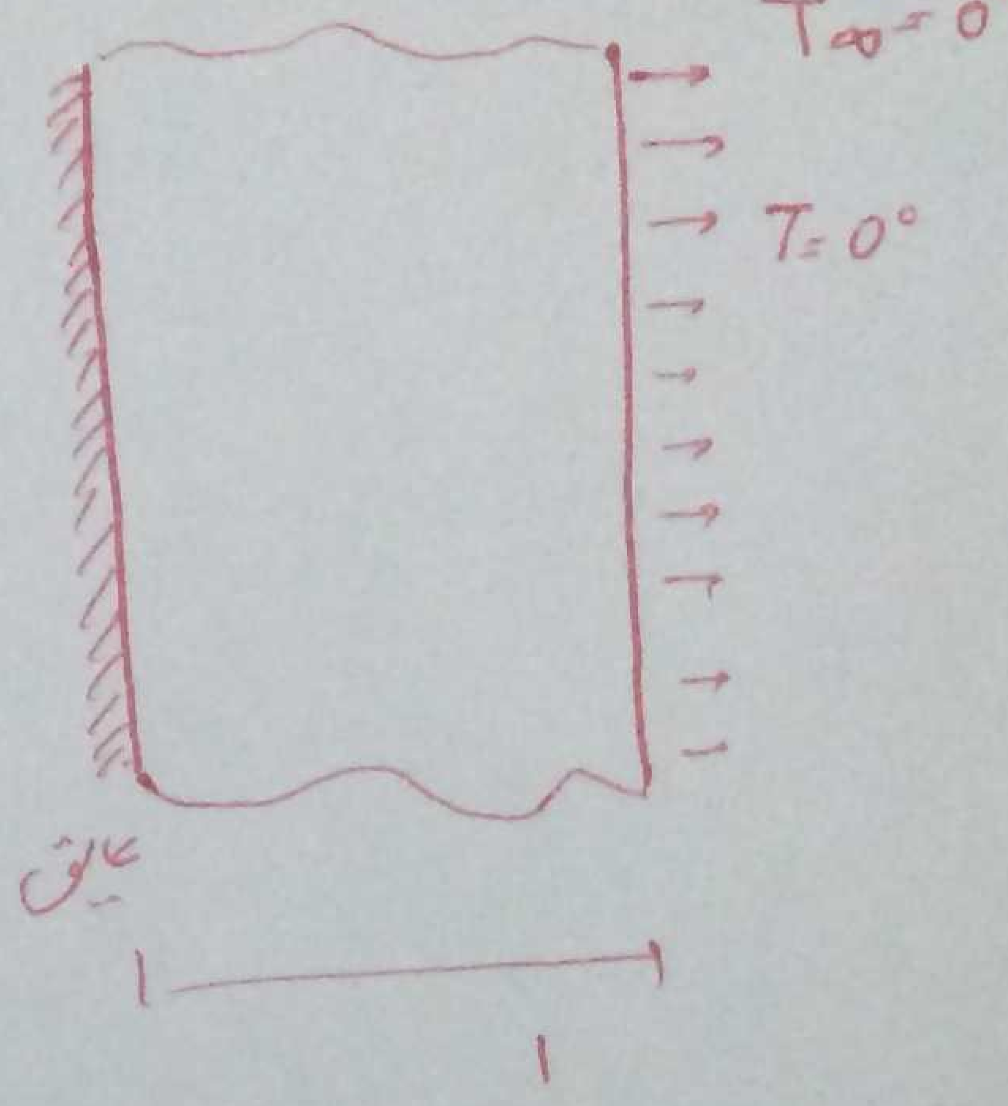
$$\begin{cases} u_t = C^2 u_{xx} & 0 \leq x \leq 1,\ t \geq 0 \\ u_x(0,t) = 0 \\ u_x(1,t) + h u(1,t) = 0 \\ u(x,0) = T_0 \end{cases}$$



از جداسازی متغیرها استفاده می‌کنیم

$$u(x,t) = F(x)G(t) \longrightarrow F\dot{G} = C^2 F'' G \longrightarrow \frac{1}{C^2}\frac{\dot{G}}{G} = \frac{F''}{F} = \begin{cases} k^2 \\ 0 \\ -k^2 \end{cases}$$

k^2، 0 منجر به جواب مطلوب نمی‌شوند لذا

$$\frac{F''}{F} = -k^2 \longrightarrow F'' + k^2 F = 0 \longrightarrow F(x) = A\cos(kx) + B\sin(kx)$$

$$\rightarrow u_x(0,t) = 0 \longrightarrow F'(x)G(t) = 0 \longrightarrow F'(x) = 0 \longrightarrow -Ak\sin(kx) + Bk\cos(kx)\Big|_{x=0} = 0$$

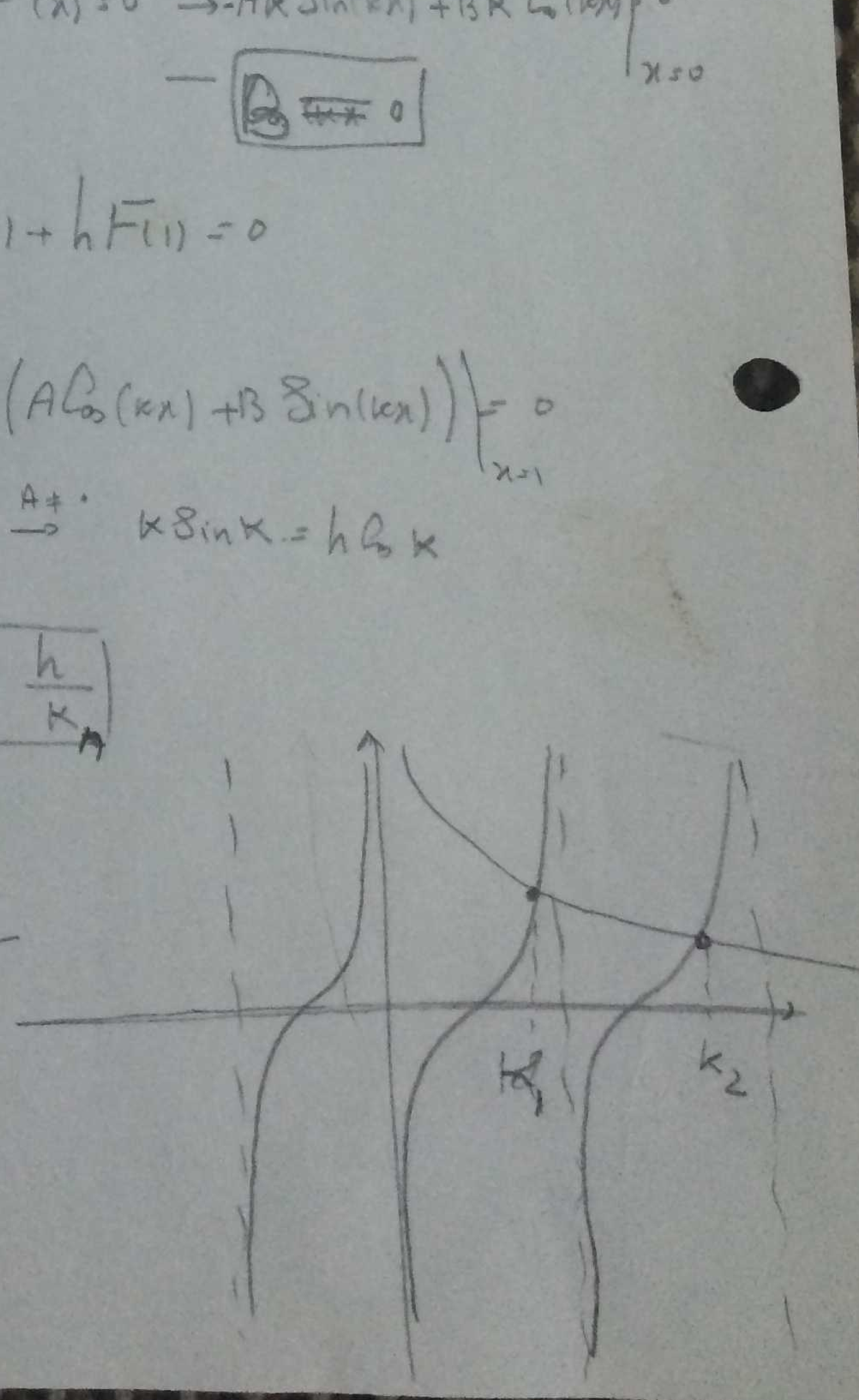
$$\longrightarrow \boxed{B = 0}$$

$$u_x(1,t) + h u(1,t) = 0 \longrightarrow F'(1) + hF(1) = 0$$

$$\Rightarrow -Ak\sin(kx) + Bk\cos(kx) + h\left(A\cos(kx) + B\sin(kx)\right)\Big|_{x=1} = 0$$

$$-Ak\sin(k) + hA\cos(k) = 0 \xrightarrow{A \neq 0} k\sin k = h\cos k$$

$$\rightarrow \frac{\sin k}{\cos k} = \frac{h}{k} \longrightarrow \boxed{\tan k_n = \frac{h}{k_n}}$$



k_n ها ریشه‌های این معادله هستند

که نامتناهی‌اند و بنابراین $n \to \infty$، $k_n \to \infty$ از تعمیم قضیه اشتورم لیوویل

می‌توان گفت که $F_n(x) = A_n \cos k_n x$ توابع ویژه هستند.

$$\frac{1}{c^2}\frac{\dot G}{G} = -k_n^2 \longrightarrow \dot G + c^2 k_n^2 G = 0 \longrightarrow G(t) = D e^{-k_n^2 c^2 t}$$

$$\longrightarrow u(x,t) = \sum_{n=1}^{\infty} E_n e^{-k_n^2 c^2 t} \cos k_n x$$

$$u(x,0) = T_0 = \sum_{n=1}^{\infty} E_n \cos k_n x$$

$$\int_0^1 T_0 \cos k_m x = E_n \int_0^1 \cos^2 k_n x \, dx$$

$$E_n = \frac{T_0 \int_0^1 \cos k_m x}{\int_0^1 \cos^2 k_m x \, dx} = \frac{T_0 \frac{1}{k_m} \sin k_m x \Big|_0^1}{\int_0^1 \frac{1 + \cos 2k_m x}{2} \, dx}$$

$$= \frac{\frac{T_0}{k_m} \sin k_m}{\frac{1}{2} x \Big|_0^1 + \frac{1}{2} \frac{1}{2k_m} \sin 2k_m \Big|_0^1}$$

$$E_n = \frac{\frac{T_0}{k_m} \sin k_m}{\frac{1}{2} + \frac{1}{4k_m} \sin 2k_m} = \boxed{\frac{4T_0 \sin k_m}{2k_m + \sin 2k_m}}$$

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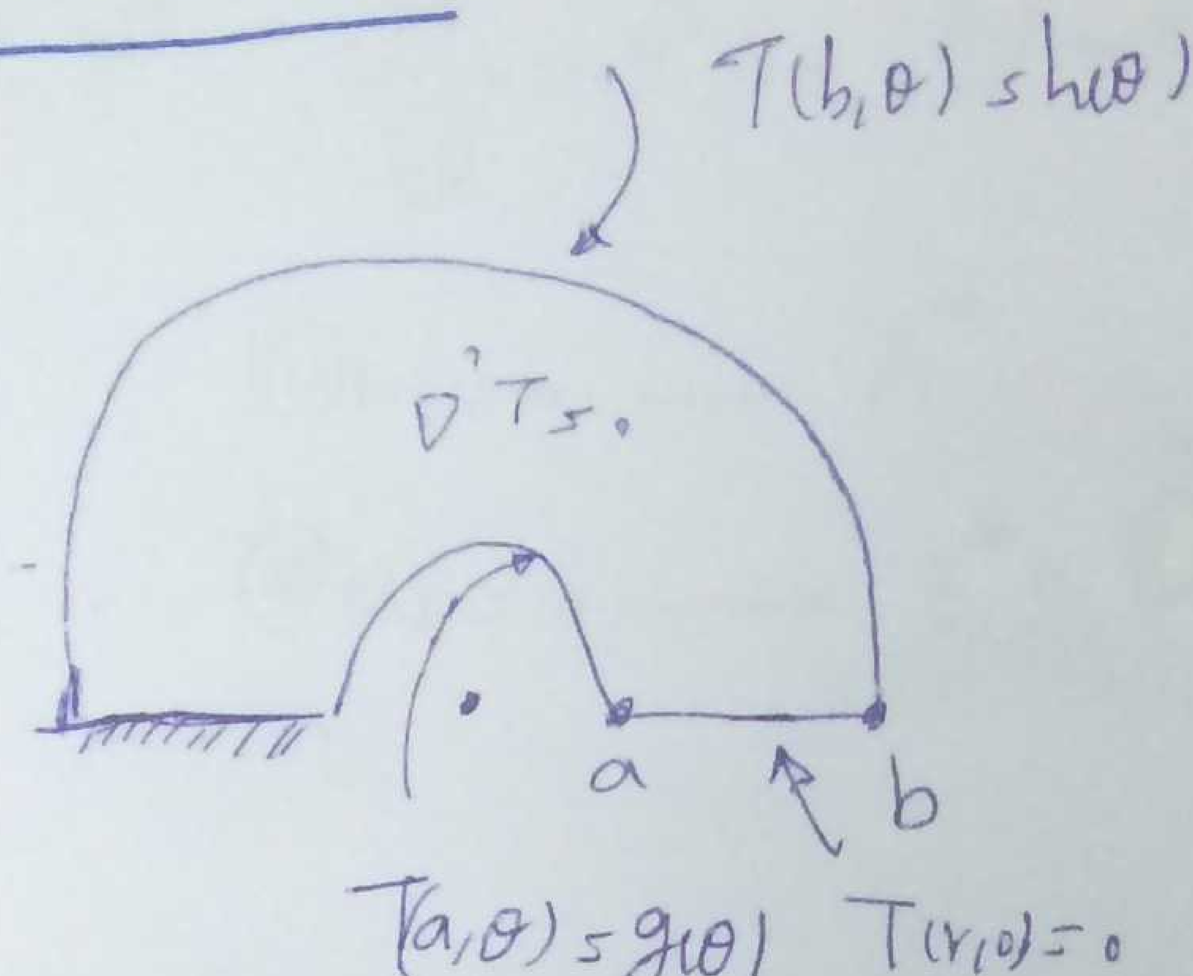

$$\begin{cases} \nabla^2 T = 0 = \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial T}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial}{\partial \theta^2}(T) = 0 \\ T(a,\theta) = g(\theta) \\ T(b,\theta) = h(\theta) \\ T(r,0) = 0 \\ \dfrac{\partial T}{\partial \theta}(r,\pi) = 0 \end{cases}$$

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$g(\theta) = 0$

$h(\theta) = \sin 2\theta$

از جداسازی متغیرها استفاده می‌کنیم:

$T(r,\theta) = R(r)\,\Theta(\theta)$

$$\longrightarrow \frac{1}{r}\frac{\partial}{\partial r}(rR'\Theta) + \frac{1}{r^2}(R\Theta'') = 0$$

$$\xrightarrow[\times r^2]{\div R\Theta} \frac{r}{R}\frac{\partial}{\partial r}(rR') + \frac{\Theta''}{\Theta} = 0$$

$$\frac{r}{R}\frac{\partial}{\partial r}(rR') = -\frac{\Theta''}{\Theta} = \begin{cases} k^2 \\ 0 \\ -k^2 \end{cases}$$

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$$\therefore -k^2 \longrightarrow \frac{-\Theta''}{\Theta} = -k^2 \longrightarrow \Theta'' - k^2\Theta = 0 \Rightarrow \Theta = Ae^{-k\theta} + Be^{k\theta}$$

۲

$$T(r,0) = 0 \rightarrow R(r)\,\Theta(0) = 0 \longrightarrow \Theta(0) = 0 \longrightarrow A + B = 0 \quad \text{I}$$

$$\frac{\partial T}{\partial \theta}(r,\pi) = 0 \rightarrow \frac{\partial}{\partial \theta}(R\Theta)\Big|_{\theta=\pi} = 0 \longrightarrow \Theta'(\pi) = 0 \rightarrow -Ake^{-k\pi} + Bke^{k\pi} = 0$$

$$-Ae^{-k\pi} + Be^{k\pi} = 0 \quad \text{II}$$

$$\text{I, II} \longrightarrow A = B = 0$$

این جواب مناسب نمی‌دهد

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$\therefore 0$

$$-\frac{\Theta''}{\Theta} = 0 \longrightarrow \Theta'' = 0, \quad \Theta = A\theta + B = 0 \longrightarrow \begin{array}{l} \Theta(0) = 0 \longrightarrow B = 0 \\ \Theta'(\pi) = 0 \longrightarrow A = 0 \end{array} \quad A = B = 0$$

IV ص

$\therefore K^2 \quad \frac{-\Theta''}{\Theta} = K^2 \rightarrow \Theta'' + K^2\Theta = 0 \longrightarrow \Theta(\theta) = A^* \cos K\theta + B^* \sin K\theta$

$\Theta(0) = 0 \longrightarrow A^* = 0$

$\Theta'(\pi) = 0 \longrightarrow B^* K \cos K\theta \Big|_{\theta=\pi} = 0 \rightarrow \cos K\pi = 0 \longrightarrow K\pi = \frac{2n-1}{2}\pi$

$\boxed{K_n = \frac{2n-1}{2}, \quad n = 1, 2, 3, \ldots}$

$\longrightarrow \Theta(\theta) = B_n^* \sin \frac{2n-1}{2}\theta$ F

$\frac{r}{R}\frac{d}{dr}(rR') = +K_n^2 \rightarrow \frac{r}{R}(R' + rR'') = K_n^2$

$r^2R'' + rR' - K_n^2 R = 0 \rightarrow$ معادله اویلر

$R(r) = r^\alpha \Rightarrow$

$R' = \alpha r^{\alpha-1}$

$R'' = \alpha(\alpha-1) r^{\alpha-2}$

$\longrightarrow r^2\alpha(\alpha-1)r^{\alpha-2} + r\alpha r^{\alpha-1} - K_n^2 r^\alpha = 0$

$(\alpha^2 - \alpha + \alpha - K_n^2) r^\alpha = 0 \rightarrow \alpha^2 - K_n^2 = 0, \quad \alpha = \pm K_n$

$R_n(r) = Cr^{\left(\frac{2n-1}{2}\right)} + Dr^{-\left(\frac{2n-1}{2}\right)}$ F

$T_n(r,\theta) = R_n(r)\,\Theta_n(\theta) = \left(C_n^* r^{\left(\frac{2n-1}{2}\right)} + D_n^* r^{-\left(\frac{2n-1}{2}\right)}\right) \sin \frac{2n-1}{2}\theta$

$\frac{2n-1}{2} = \lambda_n$

$T(r,\theta) = \sum_{n=1}^{\infty} \left(C_n^* r^{\lambda_n} + D_n^* r^{-\lambda_n}\right) \sin \lambda_n \theta$

برای بدست آوردن C_n^* ، D_n^* از شرایط مرزی غیرهمگن استفاده می‌شود:

$T(a,\theta) = \sum_{n=1}^{\infty} \left(C_n^* a^{\lambda_n} + D_n^* a^{-\lambda_n}\right) \sin \lambda_n \theta = g(\theta)$ F

$C_n^* a^{\lambda_n} + D_n^* a^{-\lambda_n} = \frac{2}{\pi}\int_0^\pi g(\theta) \sin \lambda_n \theta \, d\theta$ I

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$$T(b,\theta) = \sum_{n=1}^{\infty} \left(C_n^* b^{\lambda_n} + D_n^* b^{-\lambda_n}\right) \sin\lambda_n\theta = h(\theta)$$

$$\boxed{\left(C_n^* b^{\lambda_n} + D_n^* b^{-\lambda_n}\right) = \frac{2}{\pi}\int_0^{\pi} h(\theta) \sin\lambda_n\theta \, d\theta} \quad \text{II}$$

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$$\text{I, II} \longrightarrow C_n^*, D_n^* = \checkmark$$

$$g(\theta) = 0 \longrightarrow C_n^* a^{\lambda_n} + D_n^* a^{-\lambda_n} = 0$$

$$g(\theta) = \sin 2\theta \longrightarrow C_n^* b^{\lambda_n} + D_n^* b^{-\lambda_n} = \frac{2}{\pi}\int_0^{\pi} \sin 2\theta \sin\lambda_n\theta \, d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi} \left(\cos(2-\lambda_n)\theta - \cos(2+\lambda_n)\theta\right) d\theta$$

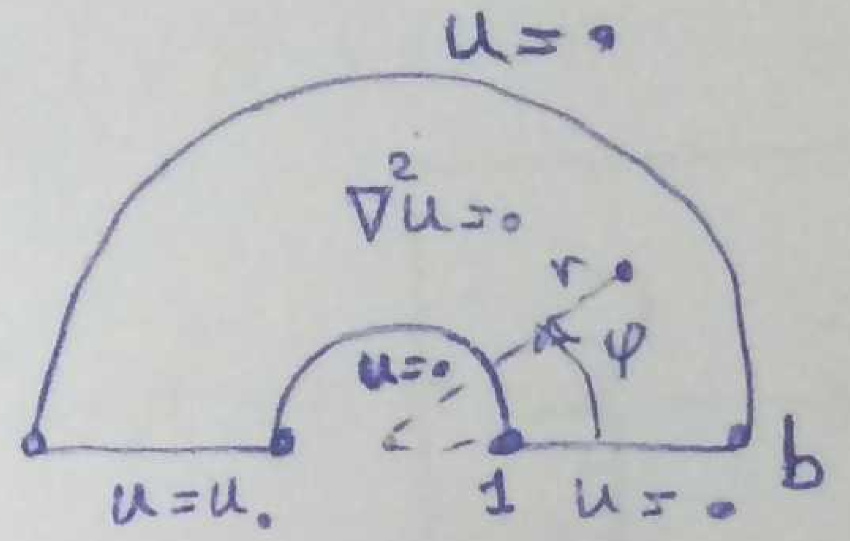
$$= \frac{2}{\pi}\left[\frac{\sin(2-\lambda_n)\theta}{2-\lambda_n} - \frac{\sin(2+\lambda_n)\theta}{2+\lambda_n}\right]_0^{\pi}$$

$$2-\lambda_n = 2 - \frac{2n-1}{2} = \frac{5-2n}{2}$$

$$2+\lambda_n = 2 + \frac{2n-1}{2} = \boxed{\frac{2n+3}{2}}$$

$$\frac{2}{\pi}\left[\frac{\pi \sin(2-\lambda_n)\pi}{(2-\lambda_n)\pi} - \frac{\pi \sin(2+\lambda_n)\pi}{(2+\lambda_n)\pi}\right]$$

مسئله: حل معادله لاپلاس در فضای



$$\begin{cases} \nabla^2 u = 0 & 1<r<b,\ 0<\varphi<\pi \\ u(r,0)=0 & 1<\rho<b \\ u(r,\pi)=u_0 & \\ u(1,\varphi)=0 & 0<\varphi<\pi \\ u(b,\varphi)=0 & \end{cases}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}=0$$

$$u(r,\varphi)=R(r)\Phi(\varphi) \longrightarrow \frac{1}{r}\frac{\partial}{\partial r}(rR'\Phi)+\frac{1}{r^2}R\Phi''=0$$

$$\xrightarrow{\div R\phi} \frac{1}{rR}\frac{d}{dr}(rR')+\frac{1}{r^2}\frac{\Phi''}{\Phi}=0 \longrightarrow \frac{r}{R}\frac{d}{dr}(rR')+\frac{\Phi''}{\Phi}=0$$

$$\longrightarrow \frac{r}{R}\frac{d}{dr}(rR')=-\frac{\Phi''}{\Phi}=\begin{cases} k^2 \\ 0 \\ -k^2 \end{cases} \quad \text{I}$$

چون در راستای شعاعی شرایط مرزی همگن است بنابراین

$$\therefore k^2 \quad \frac{r}{R}\frac{d}{dr}(rR')=k^2 \longrightarrow r[R'+rR'']-k^2R=0$$

$$r^2R''+rR'-k^2R=0 \longrightarrow R(r)=r^\alpha \longrightarrow R(r)=A^*r^{-k}+B^*r^{k}$$

$$\xrightarrow{B.C.} u(1,\varphi)=R(1)\Phi(\varphi)=0 \longrightarrow R(1)=0 \longrightarrow \begin{cases} A^*+B^*=0 \\ A^*b^{-k}+B^*b^{k}=0 \end{cases}$$

$$u(b,\varphi)=R(b)\Phi(\varphi)=0 \longrightarrow R(b)=0$$

بنابراین $A^*=B^*=0$ لذا $R(r)=0$ و پاسخ نامناسب بدیهی می‌شود.

$$\therefore 0 \quad \frac{r}{R}\frac{d}{dr}(rR')=0 \longrightarrow \frac{d}{dr}(rR')=0,\quad rR'=C^*$$

$$R'=\frac{C^*}{r} \longrightarrow R(r)=C^*\ln r+D^*$$

B.C.

$$R(1)=0 \longrightarrow C^*\ln 1+D^*=0 \longrightarrow D^*=0$$

$$R(b)=0 \longrightarrow C^*\ln b=0 \longrightarrow C^*=0$$

$\longrightarrow$ پاسخ نامناسب بدیهی می‌شود.

$$\frac{r}{R}\frac{d}{dr}(rR') = -K^2 \longrightarrow r(R' + rR'') + K^2 R = 0 \rightarrow$$

$$r^2 R'' + rR' + K^2 R = 0 \longrightarrow \text{II}$$

یادآوری

if A, B, C are Constants, the differential equation $Ax^2y'' + Bxy' + Cy = 0$ is called Cauchy-Euler equation:

With substitution, $x = e^s$ it can be transformed into the Constant Coefficient differential equation:

$$\boxed{A\frac{d^2y}{ds^2} + (B-A)\frac{dy}{ds} + Cy = 0}$$

$$r = e^s \rightarrow dr = e^s ds \rightarrow \frac{ds}{dr} = e^{-s}$$

$$\rightarrow R' = \frac{dR}{dr} = \frac{dR}{ds}\cdot\frac{ds}{dr} = \frac{dR}{ds}e^{-s}$$

$$R'' = \frac{d^2R}{dr^2} = \frac{d}{dr}\left(\frac{dR}{dr}\right) = \frac{d}{ds}\left(\frac{dR}{dr}\right)\frac{ds}{dr} = \frac{d}{ds}\left(e^{-s}\frac{dR}{ds}\right)e^{-s}$$

$$= \left[-e^{-s}\frac{dR}{ds} + e^{-s}\frac{d^2R}{ds^2}\right]e^{-s} = e^{-2s}\left[\frac{d^2R}{ds^2} - \frac{dR}{ds}\right]$$

در معادله II قرار می‌دهیم:

$$\Rightarrow \cancel{e^{2s} \times e^{-2s}}^{\,1}\left[\frac{d^2R}{ds^2} - \frac{dR}{ds}\right] + \cancel{e^{s}}\left[\frac{dR}{ds}\right]\cancel{e^{-s}}^{\,1} + K^2R = 0$$

$$\frac{d^2R}{ds^2} - \frac{dR}{ds} + \frac{dR}{ds} + K^2R = 0 \longrightarrow \boxed{\frac{d^2R}{ds^2} + K^2R = 0}$$

$$R(s) = A\cos ks + B\sin ks$$

$$R(r) = A\cos(k\ln r) + B\sin(k\ln r)$$

B.C

$$R(1)=0 \longrightarrow A\cos(0)+B\sin(0)=0 \longrightarrow A=0$$

$$R(b)=0 \longrightarrow B\sin(k\ln b)=0 \xrightarrow{B\neq 0} \sin(k\ln b)=0$$

$$k\ln(b)=n\pi \longrightarrow k_n=\frac{n\pi}{\ln b}$$

$$R_n(r)=B_n\sin\left(\frac{n\pi}{\ln b}\ln r\right)=B_n\sin(k_n\ln r)$$

مورد ۲
قسمت دوم

I

$$-\frac{\Phi''}{\Phi}=-k_n^2 \longrightarrow \Phi''-k_n^2\Phi=0 \longrightarrow \Phi_n(\varphi)=C_n\cosh k_n\varphi+D_n\sinh k_n\varphi$$

$$\rightarrow u_n(r,\varphi)=R_n(r)\Phi_n(\varphi)=(E_n\cosh k_n\varphi+F_n\sinh k_n\varphi)\sin(k_n\ln r)$$

$$u(r,\varphi)=\sum_{n=1}^{\infty}(E_n\cosh k_n\varphi+F_n\sinh k_n\varphi)\sin(k_n\ln r)$$

$$u(r,0)=\sum E_n\sin(k_n\ln r)=0 \longrightarrow E_n=0$$

$$u(r,\pi)=u_0=\sum F_n\sinh k_n\pi\,\sin(k_n\ln r) \quad \text{III}$$

برای محاسبه F_n ، می‌دانیم $\{\sin(k_n\ln r)\}$ مجموعه، چون توابع ویژه‌ی معادله اشتورم‌لیوویل هستند لذا تشکیل یک مجموعه متعامد نسبت به تابع وزنی $\frac{1}{r}$ می‌دهند. طرفین معادله III را در $\frac{1}{r}\sin(k_m\ln r)$ ضرب کرده و از $r=1$ تا $r=b$ انتگرال می‌گیریم:

$$\begin{cases} r^2R''+rR'+k^2R=0 \\ (rR'(r))'+\frac{k^2}{r}R=0 \end{cases} \rightarrow \left(\begin{matrix}\text{تابع وزنی} \\ p(r)=\frac{1}{r}\end{matrix}\right)$$

$$\Rightarrow \int_1^r \frac{1}{r}u_0\sin(k_m\ln r)\,dr=\sum_{n=1}^{\infty}\int_1^r F_n\sinh k_n\pi\,\sin(k_n\ln r)\sin(k_m\ln r)\frac{1}{r}\,dr$$

صفحة (٢٢)

$$u_0\int_1^b \frac{1}{r}\sin(k_m \ln r)\,dr = F_m \sinh k_m\pi \int_1^b \frac{1}{r}\sin^2(k_m \ln r)\,dr$$

$$F_m\frac{\sinh k_m\pi}{u_0} = \frac{\int_1^b \frac{1}{r}\sin(k_m\ln r)\,dr}{\int_1^b \frac{1}{r}\sin^2(k_m\ln r)\,dr}$$

$$\int_1^b \frac{1}{r}\sin^2(k_m\ln r)\,dr = \int_1^b \frac{1}{2r}\left[1-\cos(2k_m\ln r)\right]dr = \frac{1}{2}\left[\int_1^b \frac{dr}{r} - \int_1^b \frac{1}{r}\cos(2k_m\ln r)\,dr\right]$$

$\ln r = u$

$\frac{dr}{r} = du$

$$= \frac{1}{2}\left[\ln r\Big|_1^b - \int_0^{\ln b}\cos(2k_m u)\,du\right] =$$

$$= \frac{1}{2}\left[\ln b - \frac{1}{2k_m}\sin(2k_m u)\Big|_0^{\ln b}\right]$$

$$= \frac{1}{2}\left[\ln b - \frac{1}{2k_m}\sin(2k_m\ln b)\right] \quad , \quad k_m\ln b = n\pi$$

$\sin(2n\pi) = 0$

$$= \boxed{\frac{\ln b}{2}}$$

$$\int_1^b \frac{1}{r}\sin k_m\ln r\,dr = \int_0^{\ln b}\sin k_m u\,du = -\frac{1}{k_m}\cos k_m u\Big|_0^{\ln b} = -\frac{1}{k_m}\left[\cos k_m\ln b - \cos 0\right]$$

$u = \ln r$

$\frac{dr}{r} = du$

$$= -\frac{1}{k_m}\left[\cos m\pi - \cos 0\right] = \frac{+\ln b}{m\pi}\left[1-(-1)^m\right]$$

$$F_m\frac{\sinh k_m\pi}{u_0} = \frac{\frac{\ln b}{m\pi}\left[1-(-1)^m\right]}{\frac{\ln b}{2}} = \frac{2}{m\pi}\left[1-(-1)^m\right]$$

$$F_m = \frac{2u_0\left[1-(-1)^m\right]}{m\pi\sinh k_m\pi} \Rightarrow$$

$$u(r,\varphi) = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n}\,\frac{\sinh(k_n\varphi)}{\sinh(k_n\pi)}\sin(k_n\ln r)$$