CHAPTER 3

EXACT ONE-DIMENSIONAL SOLUTIONS

3.1 Introduction

- Temperature solution depends on velocity
- Velocity is governed by non-linear Navier-Stokes eqs.
- Exact solution are based on simplifications governing equations

3.2 Simplification of the Governing Equations

Simplifying assumptions:

(1) Laminar flow

(2) Parallel streamlines

$v = 0$ (3.1)

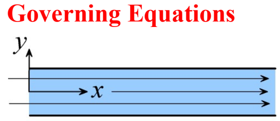

Fig. 3.1

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(3.1) into continuity for 2-D, constant density fluid:

$$\frac{\partial u}{\partial x} = 0 \text{ , everywhere} \tag{3.2}$$

$\therefore$

$$\frac{\partial^2 u}{\partial x^2} = 0 \tag{3.3}$$

(3) Negligible axial variation of temperature

$$\frac{\partial T}{\partial x} = 0 \text{ , everywhere} \tag{3.4}$$

(3.4) is valid under certain conditions. It follows that

$$\frac{\partial^2 T}{\partial x^2} = 0 \tag{3.5}$$

(4) Constant properties: velocity and temperature fields are *uncoupled* (Table 2.1, white box)

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TABLE 2.1

Basic law	No. of Equations	Unknowns							
Energy	1	T	u	v	w		ρ	μ	k
Continuity	1		u	v	w		ρ		
Momentum	3		u	v	w	p	ρ	μ	
Equation of State	1	T				p	ρ		
Viscosity relation $\mu = \mu(p,T)$	1	T				p		μ	
Conductivity relation $k = k(p,T)$	1	T				p			k

Similar results are obtained for certain rotating flows.

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Fig. 3.2:

- Shaft rotates inside sleeve
- Streamlines are concentric circles
- Axisymmetric conditions, no axial variations

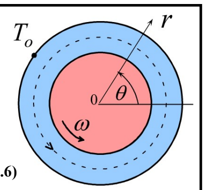


Fig. 3.2

$$\therefore \quad \frac{\partial T}{\partial \theta} = 0 \quad (3.6)$$

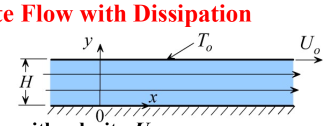
$$\frac{\partial^2 T}{\partial \theta^2} = 0 \quad (3.7)$$

3.3 Exact Solutions

3.3.1 Couette Flow

- Flow between parallel plate
- Motion due to pressure drop and/or moving plate
- Channel is infinitely long

Example 3.1: Couette Flow with Dissipation



- Very large parallel plates
- Incompressible fluid
- Upper plate at T_o moves with velocity U_o
- Insulate lower plate
- Account for dissipation
- Laminar flow, no gravity, no pressure drop
- Determine temperature distribution

(1) Observations

- Plate sets fluid in motion
- No axial variation of flow
- Incompressible fluid
- Cartesian geometry

(2) Problem Definition.

Determine the velocity and temperature distribution

(3) Solution Plan

- Find flow field, apply continuity and Navier-Stokes equations
- Apply the energy to determine the temperature distribution

(4) Plan Execution

(i) Assumptions

- Steady state
- Laminar flow
- Constant properties
- Infinite plates
- No end effects
- Uniform pressure
- No gravity

(ii) Analysis

Start with the energy equation

$$\rho c_P\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=$$

$$k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\mu\Phi \qquad \text{(3.19b)}$$

Φ is dissipation

$$\Phi=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+$$

$$\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2\right]-$$

$$\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2 \qquad \text{(3.17)}$$

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Need u, v and w. Apply continuity and the Navier-Stokes equations

Continuity

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}++v\frac{\partial \rho}{\partial y}+w\frac{\partial \rho}{\partial z}+\rho\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right]=0 \qquad \text{(2.2b)}$$

Constant density

$$\frac{\partial \rho}{\partial t}=\frac{\partial \rho}{\partial x}=\frac{\partial \rho}{\partial y}=\frac{\partial \rho}{\partial z}=0 \qquad \text{(a)}$$

Infinite plates

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial z}=w=0 \qquad \text{(b)}$$

(a) and (b) into (2.2b)

$$\frac{\partial v}{\partial y}=0 \qquad \text{(c)}$$

Integrate (c)

$$v=f(x) \qquad \text{(d)}$$

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$f(x)$ is "constant" of integration

Apply the no-slip condition

$$v(x,0)=0 \qquad \text{(e)}$$

(d) and (e) give

$$f(x)=0$$

Substitute into (d)

$$v=0 \qquad \text{(f)}$$

$\therefore$ Streamlines are parallel

To determine u we apply the Navier-Stokes eqs.

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=$$

$$\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \qquad \text{(2.10x)}$$

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Simplify:

Steady state $$\frac{\partial u}{\partial t} = 0 \qquad \text{(g)}$$

No gravity $$g_x = 0 \qquad \text{(h)}$$

Negligible axial pressure variation
$$\frac{\partial p}{\partial x} = 0 \qquad \text{(i)}$$

(b) and (f)-(i) into (2.10x) gives
$$\frac{d^2 u}{dy^2} = 0 \qquad \text{(j)}$$

Solution to (j) is
$$u = C_1 y + C_2 \qquad \text{(k)}$$

Boundary conditions
$$u(0) = 0 \ \text{and} \ u(H) = U_o \qquad \text{(l)}$$

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(k) and (I) give
$$C_1 = U_o \ \text{and} \ C_2 = 0 \qquad \text{(m)}$$

(m) into (k)
$$\frac{u}{U_o} = \frac{y}{H} \qquad \text{(3.8)}$$

Dissipation: (b) and (f) into (2.17)
$$\Phi = \left(\frac{\partial u}{\partial y}\right)^2 \qquad \text{(n)}$$

Use (3.8) into (n)
$$\Phi = \frac{U_o^2}{H^2} \qquad \text{(o)}$$

Steady state: $\partial T / \partial t = 0$

Infinite plates at uniform temperature:
$$\frac{\partial T}{\partial x} = \frac{\partial^2 T}{\partial x^2} = 0$$

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Use above, (b), (f) and (o) into energy (2.10b)
$$k\frac{d^2 T}{dy^2} + \mu\frac{U_o^2}{H^2} = 0 \qquad \text{(p)}$$

Integrate
$$T = -\frac{\mu U_o^2}{2kH^2} y^2 + C_3 y + C_4 \qquad \text{(q)}$$

B.C.
$$-k\frac{dT(0)}{dy} = 0 \quad \text{and} \quad T(H) = T_o \qquad \text{(r)}$$

B.C. and solution (q) give
$$C_3 = 0 \ \text{and} \ C_4 = T_o + \frac{\mu U_o^2}{2k} \qquad \text{(s)}$$

(s) into (q)
$$\frac{T - T_o}{\frac{\mu U_o^2}{k}} = \frac{1}{2}\left(1 - \frac{y^2}{H^2}\right) \qquad \text{(3.9)}$$

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Fourier's law gives heat flux at $y = H$

$$q''(H) = -k\frac{dT(H)}{dx}$$

(3.9) into the above

$$q''(H) = \frac{\mu U_o^2}{H} \qquad (3.10)$$

(iii) Checking

Dimensional check: Each term in (3.8) and (3.9) is dimensionless. Units of (3.10) is W/m^2

Differential equation check: Velocity solution (3.8) satisfies (j) and temperature solution (3.9) satisfies (p)

Boundary conditions check: Solution (3.8) satisfies B.C. (l), temperature solution (3.9) satisfies B.C. (r)

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Limiting check: (i) Stationary upper plate: no fluid motion. Set $U_o = 0$ in (3.8) gives $u(y) = 0$

(ii) Stationary upper plate: no dissipation, uniform temperature T_o, no surface flux. Set $U_o = 0$ in (o), (3.9) and (3.10) gives $\Phi = 0$, $T(y) = T_o$ and $q''(H) = 0$

(iii) Inviscid fluid: no dissipation, uniform temperature T_o. Set $\mu = 0$ in (3.9) gives $T(y) = T_o$

(iv) Global conservation of energy: Frictional energy is conducted through moving plate:

W = Friction work by plate

$q''(H)$ = Heat conducted through plate

$$W = \tau(H)U_o \qquad (t)$$

where

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$\tau(H)$ = shearing stress

$$\tau(H) = \mu\frac{du(H)}{dy} \qquad (u)$$

(3.8) into (u)

$$\tau(H) = \mu\frac{U_o}{H} \qquad (v)$$

(v) and (t)

$$W = \frac{\mu U_o^2}{H} \qquad (w)$$

(w) agrees with (3.10)

(4) Comments

- Infinite plate is key assumption. This eliminates x as a variable
- Maximum temperature: at $y = 0$ Set $y = 0$ in (3.9)

$$T(0) - T_o = \frac{\mu U_o^2}{2k}$$

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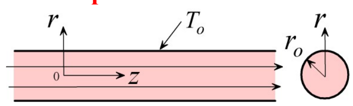
3.3.2 Hagen-Poiseuille Flow

- Problems associated with axial flow in channels
- Motion due to pressure drop
- Channel is infinitely long

Example 3.2: Flow in a Tube at Uniform Surface Temperature

- Incompressible fluid flows in a long tube
- Motion is due to pressure gradient $\partial p / \partial z$



- Surface temperature T_o
- Account for dissipation
- Assuming axisymmetric laminar flow

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- Neglecting gravity and end effects
- Determine:

[*a*] *Temperature distribution*

[*b*] *Surface heat flux*

[*c*] *Nusselt number based on* $[T(0) - T_o]$

(1) Observations

- Motion is due to pressure drop
- Long tube: No axial variation
- Incompressible fluid
- Heat generation due to dissipation
- Dissipated energy is removed by conduction at the surface
- Heat flux and heat transfer coefficient depend on temperature distribution

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- Temperature distribution depends on the velocity distribution
- Cylindrical geometry

(2) Problem Definition.

Determine the velocity and temperature distribution.

(3) Solution Plan

- Apply continuity and Navier-Stokes to determine flow field
- Apply energy equation to determine temperature distribution
- Fourier's law surface heat flux
- Equation (1.10) gives the heat transfer coefficient.

(4) Plan Execution

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(i) Assumptions

- Steady state
- Laminar flow
- Axisymmetric flow
- Constant properties
- No end effects
- Uniform surface temperature
- Negligible gravitational effect

(ii) Analysis

[a] Start with energy equation (2.24)

$$\rho c_P\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+v_z\frac{\partial T}{\partial z}\right)=$$
$$k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right]+\mu\Phi \qquad (2.24)$$

where

$$\Phi=2\left(\frac{\partial v_r}{\partial r}\right)^2+2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)^2+2\left(\frac{\partial v_z}{\partial z}\right)^2+$$
$$\left(\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)^2+\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}+\frac{\partial v_\theta}{\partial z}\right)^2+\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)^2 \qquad (2.25)$$

- Need v_r, v_θ and v_z
- Flow field: use continuity and Navier-Stokes eqs.

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0 \qquad (2.4)$$

Constant ρ

$$\frac{\partial \rho}{\partial t}=\frac{\partial \rho}{\partial r}=\frac{\partial \rho}{\partial \theta}=\frac{\partial \rho}{\partial z}=0 \qquad (a)$$

Axisymmetric flow

$$v_\theta=\frac{\partial}{\partial \theta}=0 \qquad (b)$$

Long tube, no end effects

$$\frac{\partial}{\partial z}=0 \qquad (c)$$

(a)-(c) into (2.4)

$$\frac{d}{dr}(r v_r)=0 \qquad (d)$$

Integrate

$$r v_r=f(z) \qquad (e)$$

$f(z)$ is "constant" of integration. Use the no-slip B.C.

$$v(r_o,z)=0 \qquad (f)$$

(e) and (f) give

$$f(z)=0$$

Substitute into (e)

$$v_r=0 \qquad (g)$$

$\therefore$ Streamlines are parallel

v_z Determine : Navier-Stokes eq. in *z*-direction

$$\rho\left(v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}+\frac{\partial v_z}{\partial t}\right)=$$

$$\rho g_z-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right] \quad (2.11z)$$

Simplify

Steady state: $$\frac{\partial}{\partial t}=0 \quad (h)$$

No gravity: $$g_r=g_z=0 \quad (i)$$

(b), (c) and (g)-(i) into (2.11z)

$$-\frac{\partial p}{\partial z}+\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)=0 \quad (3.11)$$

$\therefore v_z$ depends on r only, rewrite (3.11)

$$\frac{\partial p}{\partial z}=\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)=g(r) \quad (j)$$

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Integrate

$$p=g(r)z+C_o \quad (k)$$

Apply Navier-Stokes in r-direction

$$\rho\left(v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^{\,2}}{r}+v_z\frac{\partial v_r}{\partial z}+\frac{\partial v_r}{\partial t}\right)=$$

$$\rho g_r-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right] \quad (2.11r)$$

(b), (g) and (i) into (2.11r)

$$\frac{\partial p}{\partial r}=0 \quad (l)$$

Integrate

$$p=f(z) \quad (m)$$

$f(z)=$ "constant" of integration

- **Equate two solutions for p: (k) and (m):**

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$$p=g(r)z+C_o=f(z) \quad (n)$$

$\therefore$

$$g(r)=C \quad (o)$$

C = constant. Use (o) into (j)

$$\frac{\partial p}{\partial z}=\mu\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)=C \quad (p)$$

Integrate

$$r\frac{dv_z}{dr}=\frac{1}{2\mu}\frac{dp}{dz}r^2+C_1$$

integrate again

$$v_z=\frac{1}{4\mu}\frac{dp}{dz}r^2+C_1\ln r+C_2 \quad (q)$$

Two B.C. on : v_z

$$\frac{dv_z(0)}{dr}=0,\quad v_z(r_o)=0 \quad (r)$$

(q) and (r) give C_1 and C_2

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$$C_1 = 0 \quad , \quad C_2 = \frac{1}{4\mu}\frac{dp}{dz} r_o^2$$

Substitute into (q)

$$v_z = \frac{1}{4\mu}\frac{dp}{dz}(r^2 - r_o^2) \tag{3.12}$$

For long tube at uniform temperature:

$$\frac{\partial T}{\partial z} = \frac{\partial^2 T}{\partial z^2} = 0 \tag{s}$$

(b), (c), (g), (h) and (s) into energy (2.24)

$$k \frac{1}{r}\frac{d}{dr}\left(r \frac{dT}{dr} \right) + \mu \Phi = 0 \tag{t}$$

(b), (c) and (g) into (2.25)

$$\Phi = \left(\frac{dv_z}{dr} \right)^2$$

Substitute velocity solution (3.11) into the above

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$$\Phi = \left(\frac{1}{2\mu}\frac{dp}{dz} \right)^2 r^2 \tag{u}$$

(u) in (t) and rearrange

$$\frac{d}{dr}\left(r \frac{dT}{dr} \right) = -\frac{1}{4k\mu}\left(\frac{dp}{dz} \right)^2 r^3 \tag{3.13}$$

Integrate

$$T = -\frac{1}{64k\mu}\left(\frac{dp}{dz} \right)^2 r^4 + C_3 \ln r + C_4 \tag{v}$$

Need two B.C.

$$\frac{dT(0)}{dr} = 0 \quad and \quad T(r_o) = T_o \tag{w}$$

(v) and (w) give

$$C_3 = 0 \quad , \quad C_4 = T_o + \frac{1}{64k\mu}\left(\frac{dp}{dz} \right)^2 r_o^4$$

Substitute into (v)

$$T = T_o + \frac{r_o^4}{64k\mu}\left(\frac{dp}{dz} \right)^2 \left(1 - \frac{r^4}{r_o^4} \right) \tag{3.14a}$$

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In dimensionless form:

$$\frac{T - T_o}{\dfrac{r_o^4}{64k\mu}\left(\dfrac{dp}{dz} \right)^2} = \left(1 - \frac{r^4}{r_o^4} \right) \tag{3.14b}$$

[b] Use Fourier's

$$q''(r_o) = -k \frac{dT(r_o)}{dr}$$

(3.14) into above

$$q''(r_o) = \frac{r_o^3}{16\mu}\left(\frac{dp}{dz} \right)^2 \tag{3.15}$$

[c] Nusselt number:

$$Nu = \frac{hD}{k} = \frac{2hr_o}{k} \tag{x}$$

(1.10) gives *h*

$$h = -\frac{k}{[T(0) - T_o]}\frac{dT(r_o)}{dr} \tag{y}$$

(3.14a) into (y)

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(z) into (x)

$$h = \frac{2k}{r_o} \quad \text{(z)}$$

$$Nu = 4 \quad (3.16)$$

(iii) Checking

Dimensional check:

- **Each term in (3.12) has units of velocity**
- **Each term in (3.14a) has units of temperature**
- **Each term in (3.15) has units of W/m²**

Differential equation check: **Velocity solution (3.12) satisfies (p) and temperature solution (3.14) satisfies (3.13)**

Boundary conditions check: **Velocity solution (3.12) satisfies B.C. (r) and temperature solution (3.14) satisfies B.C. (w)**

Limiting check:

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(i) Uniform pressure ($dp/dz = 0$): No fluid motion.
Set $dp/dz = 0$ in (3.12) gives $v_z = 0$

(ii) Uniform pressure ($dp/dz = 0$): No fluid motion, no dissipation, no surface flux. Set $dp/dz = 0$ in (3.15) gives $q''(r_o) = 0$

(iii) Global conservation of energy:

Heat leaving tube = Pump work

Pump work W for a tube of length L

$$W = (p_1 - p_2)\dot{Q} \quad \text{(z-1)}$$

p_1 = **upstream pressure**
p_2 = **downstream pressure**
$\dot{Q}$ = **flow rate**

$$\dot{Q} = 2\pi \int_0^{r_o} v_z r dr$$

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(3.12) into the above, integrate

$$\dot{Q} = -\frac{\pi}{8\mu}\frac{dp}{dz} r_o^4 \quad \text{(z-2)}$$

(z-1) and (z-2)

$$W = -\frac{\pi r_o^4}{8\mu}\frac{dp}{dz}(p_1 - p_2) \quad \text{(z-3)}$$

Work per unit area W''

$$W'' = \frac{W}{2\pi r_o L}$$

(z-3) into the above

$$W'' = -\frac{r_o^3}{16\mu}\frac{dp}{dz}\frac{(p_1 - p_2)}{L} \quad \text{(z-4)}$$

However $$\frac{(p_1 - p_2)}{L} = -\frac{dp}{dz}$$

Combine with (z-4)

$$W'' = \frac{r_o^3}{16\mu}\left(\frac{dp}{dz}\right)^2$$

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This agrees with (3.15)

(5) Comments

- **Key simplification: long tube with end effects. This is same as assuming parallel streamlines**
- **According to (3.14), maximum temperature is at center $r = 0$**
- **The Nusselt number is constant independent of Reynolds and Prandtl numbers**

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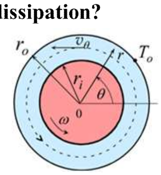
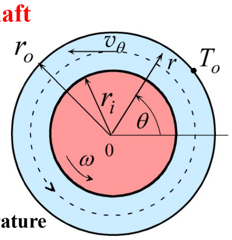
Example 3.3: Lubrication Oil Temperature in Rotating Shaft

Lubrication oil fills the clearance between a shaft and its housing. The radius of the shaft is r_i and its angular velocity is ω. The housing radius is r_0 and its temperature is T_o Assuming laminar flow and taking into consideration dissipation, determine the maximum temperature rise in the oil and the heat generated due to dissipation?



3.3.3 Rotating Flow
Example 3.3: Lubrication Oil Temperature in Rotating Shaft



- **Lubrication oil between shaft and housing**
- **Angular velocity is ω**
- **Assuming laminar flow**
- **Account for dissipation**
- **Determine the maximum temperature rise in oil**

(1) Observations

- **Fluid motion is due to shaft rotation**
- **Housing is stationary**

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- No axial variation in velocity and temperature
- No variation with angular position
- Constant ρ
- Frictional heat is removed at housing
- No heat is conducted through shaft
- Maximum temperature at shaft
- Cylindrical geometry

(2) Problem Definition.

Determine the velocity and temperature distribution of oil

(3) Solution Plan

- Apply continuity and Navier-Stokes eqs. to determine flow field
- Use energy equation to determine temperature field
- Fourier's law at the housing gives frictional heat

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(4) Plan Execution

(i) Assumptions

- Steady state
- Laminar flow
- Axisymmetric flow
- Constant properties
- No end effects
- Uniform surface temperature
- Negligible gravitational effect

(ii) Analysis

- Energy equation governs temperature

$$\rho c_P\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+v_z\frac{\partial T}{\partial z}\right)=$$

$$k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right]+\mu\Phi \qquad (2.24)$$

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where

$$\Phi=2\left(\frac{\partial v_r}{\partial r}\right)^2+2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)^2+2\left(\frac{\partial v_z}{\partial z}\right)^2+$$

$$\left(\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)^2+\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}+\frac{\partial v_\theta}{\partial z}\right)^2+\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)^2 \qquad (2.25)$$

Need flow field v_r, v_θ and v_z

- Apply continuity and Navier-Stokes to determine flow field

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0 \qquad (2.4)$$

Constant ρ

$$\frac{\partial \rho}{\partial t}=\frac{\partial \rho}{\partial r}=\frac{\partial \rho}{\partial \theta}=\frac{\partial \rho}{\partial z}=0 \qquad (a)$$

Axisymmetric flow

$$\frac{\partial}{\partial \theta}=0 \qquad (b)$$

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Long shaft:

$$v_z = \frac{\partial}{\partial z} = 0 \qquad \text{(c)}$$

(a)-(c) into (2.4)

$$\frac{d}{dr}(r v_r) = 0 \qquad \text{(d)}$$

Integrate

$$r v_r = C \qquad \text{(e)}$$

Apply B.C. to determine C

$$v_r(r_o) = 0 \qquad \text{(f)}$$

(e) and (f) give $C = 0$

Use (e)

$$v_r = 0 \qquad \text{(g)}$$

∴ Streamlines are concentric circles

Apply the Navier-Stokes to determine v_θ

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$$\rho\left(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} - \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} + \frac{\partial v_\theta}{\partial t}\right) =$$

$$\rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right] \qquad (2.11\,\theta)$$

For steady state:

$$\frac{\partial}{\partial t} = 0 \qquad \text{(h)}$$

Neglect gravity, use (b),(c), (g), (h) into (2.11 θ)

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r v_\theta)\right) = 0 \qquad (3.17)$$

Integrate

$$v_\theta = \frac{C_1}{2} r + \frac{C_2}{r} \qquad \text{(i)}$$

B.C. are

$$v_\theta(r_i) = \omega r_i \qquad v_\theta(r_o) = 0 \qquad \text{(j)}$$

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(j) gives C_1 and C_2

$$C_1 = -\frac{2\omega r_i^2}{r_o^2 - r_i^2} \qquad C_2 = \frac{\omega r_i^2 r_o^2}{r_o^2 - r_i^2} \qquad \text{(k)}$$

(k) into (i)

$$\frac{v_\theta(r)}{\omega r_i} = \frac{(r_o / r_i)^2 (r_i / r) - (r / r_i)}{(r_o / r_i)^2 - 1} \qquad (3.18)$$

Simplify energy equation (2.24) and dissipation function (2.25). Use(b), (c), (g), (h)

$$k\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) + \mu\Phi = 0 \qquad \text{(l)}$$

and

$$\Phi = \left(\frac{dv_\theta}{dr} - \frac{v_\theta}{r}\right)^2$$

(3.18) into above

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$$\Phi = \left[\frac{2\omega r_i^2}{1-(r_i/r_o)^2}\right]^2 \frac{1}{r^4} \tag{m}$$

Combine (m) and (l)

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{\mu}{k}\left[\frac{2\omega r_i^2}{1-(r_i/r_o)^2}\right]^2 \frac{1}{r^3} \tag{3.19}$$

Integrate(3.19) twice

$$T(r) = -\frac{\mu}{4k}\left[\frac{2\omega r_i^2}{1-(r_i/r_o)^2}\right]^2 \frac{1}{r^2} + C_3 \ln r + C_4 \tag{n}$$

Need two B.C.

$$T(r_o) = T_o \quad \text{and} \quad \frac{dT(r_i)}{dr} = 0 \tag{o}$$

(n) and (o) give C_3 and C_4

$$C_3 = -\frac{\mu}{2k}\left[\frac{2\omega r_i^2}{1-(r_i/r_o)^2}\right]^2 \frac{1}{r_i^2}$$

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$$C_4 = T_o + \frac{\mu}{4k}\left[\frac{2\omega r_i^2}{1-(r_i/r_o)^2}\right]^2 \left[\frac{1}{r_o^2} + \frac{2}{r_i^2}\ln r_o\right]$$

Substitute into (o)

$$T(r) = T_o + \frac{\mu}{4k}\left[\frac{2\omega r_i}{1-(r_i/r_o)^2}\right]^2 \left[(r_i/r_o)^2 - (r_i/r)^2 + 2\ln(r_o/r)\right] \tag{3.20a}$$

or

$$\frac{T(r) - T_o}{\dfrac{\mu}{4k}\left[\dfrac{2\omega r_i}{1-(r_i/r_o)^2}\right]^2} = (r_i/r_o)^2 - (r_i/r)^2 + 2\ln(r_o/r) \tag{3.20b}$$

Maximum temperature at $r = r_i$

$$T(r_i) - T_o = \frac{\mu}{4k}\left[\frac{2\omega r_i}{1-(r_i/r_o)^2}\right]^2 \left[1 + (r_i/r_o)^2 + 2\ln(r_o/r_i)\right] \tag{3.21}$$

Use Fourier's law to determine frictional energy per unit length $q'(r_o)$

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$$q'(r_o) = -2\pi r_o k \frac{dT(r_o)}{dr}$$

(3.20a) in above

$$q'(r_o) = 4\pi\mu \frac{(\omega r_i)^2}{1-(r_i/r_o)^2} \tag{3.22}$$

(iii) Checking

- Each term in solutions (3.18) and (3.20b) is dimensionless
- Equation (p) has the correct units of W/m

Differential equation check:

- Velocity solution (3.18) satisfies (3.17) and temperature solution (3.20) satisfies (3.19)

Boundary conditions check:

- Velocity solution (3.18) satisfies B.C. (j) and temperature solution (3.20) satisfies B.C. (o)

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Limiting check:

- Stationary shaft: No fluid motion. Set $\omega = 0$ in (3.18) gives $v_\theta = 0$
- Stationary shaft: No dissipation, no heat loss Set $\omega = 0$ in (3.22) gives $q'(r_o) = 0$
- Global conservation of energy:

Heat leaving housing = shaft work

Shaft work per unit length

$$W' = -2\pi r_i \tau(r_i)\omega r_i \tag{p}$$

$\tau(r_i)$ = shearing stress

$$\tau(r_i) = \mu\left[\frac{dv_\theta}{dr} - \frac{v_\theta}{r}\right]_{r=r_i} \tag{q}$$

(3.18) into the above

$$\tau(r_i) = -2\frac{(r_o/r_i)^2\,\mu\omega r_i}{(r_o/r_i)^2 - 1} \tag{r}$$

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Combining (p) and (r)

$$W' = 4\pi\mu\,\frac{(\omega r_i)^2}{1-(r_i/r_o)^2} \tag{s}$$

This is identical to surface heat transfer (3.22)

(5) Comments

- The key simplifying assumption is axisymmetry
- Temperature rise due to frictional heat increase as the clearance s decreased
- Single governing parameter: (r_i / r_o)

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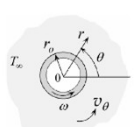
Example 3.4: A hollow shaft of outer radius, r_0 rotates with constant angular velocity, ω , while immersed in an infinite fluid at uniform temperature T_∞, Taking into consideration dissipation, determine surface heat flux. Assume incompressible laminar flow and neglect end effects.



Given

- Fluid motion is due to shaft rotation
- Axial variation in velocity and temperature are negligible for a very long shaft.
- Velocity, pressure and temperature do not vary with angular position.

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- The fluid is incompressible (constant density)
- The determination of surface temperature and heat flux requires the determination of temperature distribution in the rotating fluid.
- Use cylindrical coordinates

(2) Problem Definition(Find).

Determine velocity & temperature distribution in rotating fluid

(3) Solution Plan(Equation)

- Apply continuity and Navier-Stokes eqs. to determine flow field
- Use energy equation to determine temperature field

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(4) Plan Execution

(i) Assumptions

- Steady state
- Laminar flow
- Axisymmetric flow
- Constant properties(density, viscosity and conductivity),
- No end effects
- no angular and axial variation of velocity, pressure and temperature
- Negligible gravitational effect

(ii) Analysis

- Temperature distribution is obtained by solving the energy equation. Thus we begin the analysis with the energy equation.

$$\rho c_P\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+v_z\frac{\partial T}{\partial z}\right)= k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right]+\mu\Phi \quad (2.24)$$

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where

$$\Phi=2\left(\frac{\partial v_r}{\partial r}\right)^2+2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)^2+2\left(\frac{\partial v_z}{\partial z}\right)^2+ \left(\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)^2+\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}+\frac{\partial v_\theta}{\partial z}\right)^2+\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)^2 \quad (2.25)$$

Need flow field v_r, v_θ and v_z

- Apply continuity and Navier-Stokes to determine flow field

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0 \quad (2.4)$$

Constant ρ

$$\frac{\partial \rho}{\partial t}=\frac{\partial \rho}{\partial r}=\frac{\partial \rho}{\partial \theta}=\frac{\partial \rho}{\partial z}=0 \quad (a)$$

Axisymmetric flow

$$\frac{\partial}{\partial \theta}=0 \quad (b)$$

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Long shaft:

$$v_z = \frac{\partial}{\partial z} = 0 \qquad \text{(c)}$$

(a)-(c) into (2.4)

$$\frac{d}{dr}(r v_r) = 0 \qquad \text{(d)}$$

Integrate

$$r v_r = C \qquad \text{(e)}$$

Apply B.C. (No-slip) to determine C

$$v_r(r_o) = 0 \qquad \text{(f)}$$

(e) and (f) give $C = 0$

Use (e)

$$v_r = 0 \qquad \text{(g)}$$

∴ Streamlines are concentric circles

Apply the Navier-Stokes to determine v_θ

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$$\rho\left(v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} - \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} + \frac{\partial v_\theta}{\partial t} \right) =$$

$$\rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta} + \mu \left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) \right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right] \qquad (2.11\,\theta)$$

For steady state:

$$\frac{\partial}{\partial t} = 0 \qquad \text{(h)}$$

Neglect gravity, use (b),(c), (g), (h) into (2.11 θ)

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}(r v_\theta) \right) = 0 \qquad (3.17)$$

Integrate

$$v_\theta = \frac{C_1}{2} r + \frac{C_2}{r} \qquad \text{(i)}$$

B.C. are

$$v_\theta(r_o) = \omega r_o \qquad v_\theta(\infty) = 0 \qquad \text{(j)}$$

37

(j) gives C_1 and C_2

$$C_1 = 0 \qquad C_2 = \omega r_o^2 \qquad \text{(k)}$$

(k) into (i)

$$v_\theta(r) = r_o \omega \left(\frac{r_o}{r} \right) \qquad (3.18)$$

Simplify energy equation (2.24) and dissipation function (2.25). Use(b), (c), (g), (h)

$$k \frac{1}{r}\frac{d}{dr}\left(r \frac{dT}{dr} \right) + \mu \Phi = 0 \qquad \text{(l)}$$

and

$$\Phi = \left(\frac{d v_\theta}{dr} - \frac{v_\theta}{r} \right)^2$$

(3.18) into above

38

$$\Phi = 4\omega^2 \frac{r_o^4}{r^4} \quad \text{(m)}$$

Combine (m) and (l)

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{4\mu}{k}\frac{\omega^2 r_o^4}{r^3} \quad \text{(3.19)}$$

Integrate(3.19) twice

$$T(r) = -\frac{\mu}{k}\omega^2 r_o^4 \frac{1}{r^2} + C_3 \ln r + C_4 \quad \text{(n)}$$

Need two B.C.

$$T(\infty) = T_\infty \quad \textbf{and} \quad r \to \infty \quad T\ is\ \text{limited} \quad \text{(o)}$$

(n) and (o) give C_3 and C_4

$$C_3 = 0 \qquad C_4 = T_\infty$$

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Substitute into (o)

$$T(r) = T_\infty - \frac{\mu}{k}\omega^2 \frac{r_o^4}{r^2} \quad \text{(3.20a)}$$

Surface temperature at $r = r_o$

$$T(r_o) = T_\infty - \frac{\mu}{k}\omega^2 r_o^2 \quad \text{(3.21)}$$

Use Fourier's law to determine frictional energy per unit length $q'(r_o)$

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(3.20a) in above

$$q'(r_o) = -2\pi r_o k \frac{dT(r_o)}{dr}$$

$$q'(r_o) = -4\pi\mu(\omega r_o)^2 \quad \text{(3.22)}$$

(iii) Checking

- **Each term in solutions (3.18) and (3.20b) is dimensionless**
- **Equation (p) has the correct units of W/m**

***Differential equation check*:**

- **Velocity solution (3.18) satisfies (3.17) and temperature solution (3.20) satisfies (3.19)**

***Boundary conditions check*:**

- **Velocity solution (3.18) satisfies B.C. (j) and temperature solution (3.20) satisfies B.C. (o)**

41

***Limiting check*:**

- **Stationary shaft: No fluid motion. Set $\omega = 0$ in (3.18) gives $v_\theta = 0$**
- **Stationary shaft: No dissipation, no heat loss Set $\omega = 0$ in (3.22) gives $q'(r_o) = 0$**
- **Global conservation of energy:** Surface heat transfer rate must equal to work required to overcom friction at the shaft's surface

Shaft work per unit length

$$W' = -2\pi r_o \tau(r_o)\omega r_o \qquad \textbf{(p)}$$

$\tau(r_o)$ **= shearing stress**

$$\tau(r_o) = \mu\left[\frac{d\mathbf{v}_\theta}{dr} - \frac{\mathbf{v}_\theta}{r}\right]_{r=r_o} \qquad \textbf{(q)}$$

(3.18) into the above

$$\tau(r_o) = -2\mu\omega \qquad \textbf{(r)}$$

42

Combining (p) and (r)

$$W' = -4\pi\mu(\omega r_o)^2 \qquad \textbf{(s)}$$

This is identical to surface heat transfer (3.22)

(5) Comments

- **The key simplifying assumption is axisymmetry. This resulted in concentric streamlines with vanishing normal velocity and angular changes.**
- **Surface temperature is lowest in the entire region.**
- **Heat flow direct ion is negative.**
- **This problem was solved by specifying two conditions at infinity. If surface temperature is specified instead of fluid temperature at infinity, the solution determines T at infinity.**

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