CHAPTER 5
APPROXIMATE SOLUTIONS:
THE INTEGRAL METHOD

5.1 Introduction

- **Why approximate solution?**
- **When exact solution is:**
  - Not Available
  - Complex

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- Requires numerical integration
- Implicit

- **Approximate solution by integral method:**
  - Advantages: simple, can deal with complicating factors
  - Used in fluid flow, heat transfer, mass transfer

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5.2 Differential vs. Integral Formulation

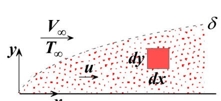

Fig. 5.1(a) Differential formulation

Differential Formulation
Conservation laws are applied to an infinitesmal element $dx \times dy$

- Result: Solutions are exact

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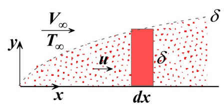

Fig. 5.1(b) Integral formulation

Integral Formulation

Conservation laws are satisfied in an average sense

- Result: Solutions are approximate

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5.3 Integral Method Approximation: Mathematical Simplification

- Number of independent variables are reduced
- Reduction in order of differential equation

5.4 Procedure

(1) Integral formulation of the basic laws

- Conservation of mass
- Conservation of momentum
- Conservation of energy

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(2) Assumed velocity and temperature profiles

- Satisfy boundary conditions
- Several possibilities
- Examples: Polynomial, linear, exponential
- Assumed profile has an unknown parameter or variable

(3) Determination of the unknown parameter or variable

- Conservation of momentum gives the unknown variable in the assumed velocity

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- Conservation of energy gives the unknown variable in the assumed temperature

5.5 Accuracy of the Integral Method

- Accuracy depends on assumed profiles
- Accuracy is not sensitive to form of profile
- Optimum profile: unknown

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5.6 Integral Formulation of the Basic Laws

5.6.1 Conservation of Mass

- Boundary layer flow
- Porous curved wall
- Conservation of mass for element $\delta \times dx$:

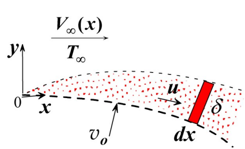

Fig. 5.2

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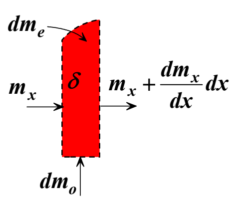

Fig. 5.3

$$m_x + dm_o + dm_e = m_x + \frac{dm_x}{dx}dx$$

$$dm_e = \frac{dm_x}{dx}dx - dm_o \quad \text{(a)}$$

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dm_e = mass from the external flow

dm_o = mass through porous wall

m_x = mass from boundary layer rate entering element at x

One-dimensional mass flow rate:

$$m = \rho VA \qquad \text{(b)}$$

Apply (b) to porous side, assume that injected is the same as external fluid

$$dm_o = \rho\, v_o\, P dx \qquad \text{(c)}$$

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P = porosity

ρ = density

Applying (b) to dy

$$dm_x = \rho u dy$$

Integrating

$$m_x = \int_0^{\delta(x)} \rho\, u dy \qquad \text{(d)}$$

(c) and (d) into (a)

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$$dm_e = \frac{d}{dx}\left[\int_0^{\delta(x)} \rho u dy\right] dx - \rho\, v_o\, P dx \qquad \text{(5.1)}$$

5.6.2 Conservation of Momentum

Apply momentum theorem to the element $\delta \times dx$

$$\sum F_x = M_x(\text{out}) - M_x(\text{in}) \qquad \text{(a)}$$

$\sum F_x$ = External x-forces on element

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M_x(in) = x-momentum of the fluid entering

M_x(out) = x-momentum of the fluid leaving

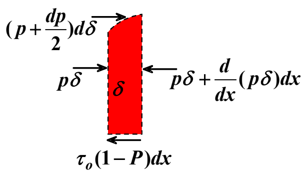

Fig. 5.4 Forces

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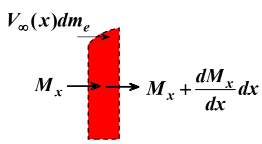

Fig. 5.4 x-Momentum

Fig. 4 and (a):

$$p\delta+\left(p+\frac{dp}{2}\right)d\delta-p\delta-\frac{d}{dx}(p\delta)dx-\tau_o(1-p)dx=$$
$$\left(M_x+\frac{dM_x}{dx}dx\right)-M_x-V_\infty(x)dm_e$$

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p = pressure

V_∞ = velocity at the edge of the boundary layer

τ_o = wall stress

$$M_x=\int_0^{\delta(x)}\rho u^2dy \quad \text{(b)}$$

and

$$\tau_o=\mu\frac{\partial u(x,0)}{\partial y} \quad \text{(c)}$$

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(b) and (c) into (a)

$$-\delta\frac{dp}{dx}-\mu(1-P)\frac{\partial u(x,0)}{\partial y}=$$
$$\frac{d}{dx}\int_0^{\delta(x)}\rho u^2 dy - V_\infty(x)\frac{d}{dx}\int_0^{\delta(x)}\rho u dy - V_\infty(x)\rho P v_o$$

- x-momentum of dm_o ?
- Shear force on slanted surface?
- (5.2) applies to laminar and turbulent flow

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- Curved surface: $V_\infty(x)$ and $p(x)$
- (5.2) is the integral formulation of conservation of momentum and mass
- (5.2) is a first order O.D.E. with x as the independent variable

Special Cases:

(i) Case 1: Incompressible fluid

$$\frac{dp}{dx}\approx\frac{dp_\infty}{dx} \tag{4.12}$$

For boundary layer flow

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$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp_\infty}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2} \tag{4.5}$$

Apply (4.5) at $y=\delta$ where $u=V_\infty$

$$\frac{dp}{dx}\approx\frac{dp_\infty}{dx}=-\rho V_\infty(x)\frac{dV_\infty}{dx} \tag{5.3}$$

(5.3) into (5.2), constant ρ

$$\delta V_\infty(x)\frac{dV_\infty}{dx}-\nu(1-P)\frac{\partial u(x,0)}{\partial y}=$$
$$\frac{d}{dx}\int_0^{\delta(x)}u^2 dy - V_\infty(x)\frac{d}{dx}\int_0^{\delta(x)}u dy - V_\infty(x)P v_o \tag{5.4}$$

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(ii) Case 2: Incompressible fluid and impermeable flat plate

Flat plate, (5.3)

$$\frac{dV_\infty}{dx} = \frac{dp}{dx} \approx \frac{dp_\infty}{dx} = 0 \qquad \text{(d)}$$

Impermeable plate

$$v_o = 0, \quad P = 0 \qquad \text{(e)}$$

(d) and (e) into (5.4)

$$\nu \frac{\partial u(x,0)}{\partial y} = V_\infty \frac{d}{dx}\int_0^{\delta(x)} u\,dy - \frac{d}{dx}\int_0^{\delta(x)} u^2 dy \qquad \text{(5.5)}$$

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5.6.3 Conservation of Energy

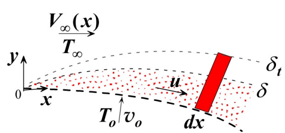

Fig. 5.5

Neglect:

(1) Changes in kinetic and potential energy

(2) Dissipation

(3) Axial conduction

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Conservation of energy element $\delta_t \times dx$

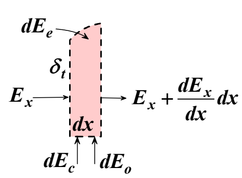

Fig. 5.6

$$E_x + dE_c + dE_o + dE_e = E_x + \frac{dE_x}{dx}dx$$

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Rearranging

$$dE_c = \frac{dE_x}{dx}dx - dE_e - dE_o$$

Fourier's law

$$dE_c = -k(1-P)\frac{\partial T(x,0)}{\partial y}dx \qquad \text{(b)}$$

Energy with mass dm_e

$$dE_e = c_p T_\infty\, dm_e$$

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Use (5.1) for dm_e

$$dE_e = c_p T_\infty \frac{d}{dx}\left[\int_0^{\delta(x)} \rho u dy\right]dx - c_p T_\infty \rho v_o P dx \qquad \text{(c)}$$

Energy of injected mass

$$dE_o = \rho c_p T_o v_o p dx \qquad \text{(d)}$$

Energy convected with fluid

$$E_x = \int_0^{\delta_t(x)} \rho c_p u T dy \qquad \text{(e)}$$

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(b)-(e) into (a)

$$-k(1-P)\frac{\partial T(x,0)}{\partial y} = \frac{d}{dx}\int_0^{\delta_t(x)} \rho c_p u T dy - c_p T_\infty \frac{d}{dx}\int_0^{\delta_t(x)} \rho u dy - \rho c_p v_o P(T_o - T_\infty) \qquad \text{(5.6)}$$

Note

(1) (5.6) is integral formulation of conservation energy

(2) (5.6) is a first order O.D.E. with *x* as the independent variable

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Special Case: Constant properties and impermeable flat plate

Simplify (5.6)

$$-\alpha\frac{\partial T(x,0)}{\partial y}=\frac{d}{dx}\int_0^{\delta_t(x)}u(T-T_\infty)dy \quad (5.7)$$

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5.7 Integral Solutions

5.7.1 Flow Field Solution:

Uniform Flow over a Semi-Infinite Plate

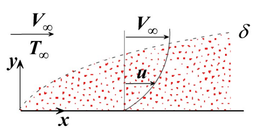

Fig. 5.7

- **Integral solution to Blasius problem**
- **Integral formulation of momentum (5.5)**

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$$\nu\frac{\partial u(x,0)}{\partial y}=V_\infty\frac{d}{dx}\int_0^{\delta(x)}udy-\frac{d}{dx}\int_0^{\delta(x)}u^2dy \quad (5.5)$$

- **Assumed velocity profile** $u(x,y)$

$$u(x,y)=\sum_{n=0}^{N}a_n(x)y^n \quad (5.8)$$

Example, a third degree polynomial

$$u(x,y)=a_0(x)+a_1(x)y+a_2(x)y^2+a_3(x)y^3 \quad \text{(a)}$$

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Boundary conditions give a_n

(1) $u(x,0)=0$ (2) $u(x,\delta)\cong V_\infty$

(3) $\dfrac{\partial u(x,\delta)}{\partial y}\cong 0$ (4) $\dfrac{\partial^2 u(x,0)}{\partial y^2}=0$

Note:

(1) B.C. (2) and (3) are approximate. Why?

(2) B.C. (4): set $y=0$ in the *x*-component of the Navier equations, (2.10x)

(3) 4 B.C. and (a) give:

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$$a_0=a_2=0 \quad , \quad a_1=\frac{3}{2}\frac{V_\infty}{\delta} \quad , \quad a_3=\frac{1}{2}\frac{V_\infty}{\delta^3}$$

$$\frac{u}{V_\infty}=\frac{3}{2}\left(\frac{y}{\delta}\right)-\frac{1}{2}\left(\frac{y}{\delta}\right)^3 \qquad (5.9)$$

Assumed velocity is in terms of a single unknown variable $\delta(x)$

(5.9) into (5.5), evaluate integrals

$$\frac{3}{2}\nu V_\infty\frac{1}{\delta}=\frac{39}{280}V_\infty^{\;2}\frac{d\delta}{dx} \qquad (b)$$

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(b) is first order O.D.E. in $\delta(x)$

$$\delta d\delta=\frac{140}{13}\frac{\nu}{U_\infty}dx$$

Integrate, use B.C. $\delta(0)=0$

$$\int_0^\delta \delta d\delta=\frac{140}{13}\frac{\nu}{U_\infty}\int_0^x dx$$

Evaluate integrals

$$\frac{\delta}{x}=\frac{\sqrt{280/13}}{\sqrt{Re_x}}=\frac{4.64}{\sqrt{Re_x}} \qquad (5.10)$$

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(5.10) into (5.9) gives $u(x, y)$

Friction coefficient C_f :Use (4.36) and (4.37a)

$$C_f = \frac{\tau_o}{\rho V_\infty^2 / 2} = \frac{\mu \frac{\partial u}{\partial y}(x,0)}{\rho V_\infty^2 / 2} = \frac{3\nu}{V_\infty \delta(x)}$$

Use (5.10) to eliminate $\delta(x)$

$$C_f = \frac{0.646}{\sqrt{Re_x}} \quad (5.11)$$

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Accuracy: Compare (5.10) and (5.11) with Blasius solution

$$\frac{\delta}{x} = \frac{5.2}{\sqrt{Re_x}}, \quad \text{Blasius solution} \quad (4.46)$$

$$C_f = \frac{0.664}{\sqrt{Re_x}}, \quad \text{Blasius solution} \quad (4.48)$$

Note:

(1) Both solutions have same form

(2) Error in $\delta(x)$ is 10.8%

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(3) Error in C_f is 2.7%

(4) Accuracy of C_f is more important than $\delta(x)$

5.7.2 Temperature Solution and Nusselt Number: Flow over a Semi-Infinite Plate

(i) Temperature Distribution

- Flat plate
- Insulated section x_o

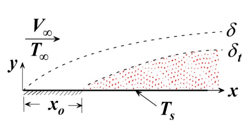

Fig. 5.8

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- **Surface at T_s**
- **Laminar, steady, two-dimensional, constant properties boundary layer flow**
- **Determine:** δ_t, $h(x)$, $Nu(x)$
- **Must determine:** $u(x,y)$ **and** $T(x,y)$ **(2)**

Integral formulation of conservation of energy

$$-\alpha\frac{\partial T(x,0)}{\partial y}=\frac{d}{dx}\int_0^{\delta(x)}\rho c_p u(T-T_\infty)dy \qquad (5.7)$$

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- **Integral Solution to** $u(x,y)$ **and** $\delta(x)$:

$$\frac{u}{V_\infty}=\frac{3}{2}\left(\frac{y}{\delta}\right)-\frac{1}{2}\left(\frac{y}{\delta}\right)^3 \qquad (5.9)$$

$$\frac{\delta}{x}=\frac{\sqrt{280/13}}{\sqrt{Re_x}}=\frac{4.64}{\sqrt{Re_x}} \qquad (5.10)$$

Assumed temperature

$$T(x,y)=\sum_{n=0}^{N}b_n(x)y^n \qquad (5.12)$$

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Let

$$T(x,y)=b_0(x)+b_1(x)y+b_2(x)y^2+b_3(x)y^3 \qquad (a)$$

Boundary conditions give $b_n(x)$

(1) $T(x,0)=T_s$

(2) $T(x,\delta_t)\cong T_\infty$

(3) $\dfrac{\partial T(x,\delta_t)}{\partial y}\cong 0$

(4) $\dfrac{\partial^2 T(x,0)}{\partial y^2}=0$

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Note

(1) B.C. (2) and (3) are approximate. Why?

(2) B.C. (4): set $y = 0$ in the *x*-component of the energy equation (2.19)

(3) Four B.C. and (a) give:

$$b_0 = T_s,\; b_1 = \frac{3}{2}(T_\infty - T_s)\frac{1}{\delta_t},\; b_2 = 0,$$

$$b_3 = -\frac{1}{2}(T_\infty - T_s)\frac{1}{\delta_t^3}$$

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Therefore

$$T(x,y) = T_s + (T_\infty - T_s)\left[\frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}\frac{y^3}{\delta_t^3}\right] \tag{5.13}$$

(5.9) and (5.13) into (5.7) and evaluating the integral

$$\frac{3}{2}\alpha\frac{T_\infty - T_s}{\delta_t} = \frac{d}{dx}\left\{(T_\infty - T_s)V_\infty\delta\left[\frac{3}{20}\left(\frac{\delta_t}{\delta}\right)^2 - \frac{3}{280}\left(\frac{\delta_t}{\delta}\right)^4\right]\right\} \tag{5.14}$$

- **(5.14) is simplified for $Pr > 1$**

$$\frac{\delta_t}{\delta} < 1\ ,\quad \text{for } Pr > 1 \tag{5.15}$$

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Last two term in (5.14):

$$\frac{3}{280}\left(\frac{\delta_t}{\delta}\right) << \frac{3}{20}\left(\frac{\delta_t}{\delta}\right)^2$$

Simplify (5.14)

$$10\frac{\alpha}{\delta_t} = V_\infty\frac{d}{dx}\left[\delta\left(\frac{\delta_t}{\delta}\right)^2\right] \tag{b}$$

Use (5.10) for δ

$$\delta = \sqrt{\frac{280}{13}}\sqrt{\frac{\nu x}{V_\infty}} \tag{c}$$

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(c) into (b)

$$\left(\frac{\delta_t}{\delta}\right)^3 + 4x\left(\frac{\delta_t}{\delta}\right)^2 \frac{d}{dx}\left(\frac{\delta_t}{\delta}\right) = \frac{13}{14}\frac{1}{Pr} \qquad \text{(d)}$$

Solve (d) for $\frac{\delta_t}{\delta}$. Let

$$r = \left(\frac{\delta_t}{\delta}\right)^3 \qquad \text{(e)}$$

(e) into (d)

$$r + \frac{4}{3}x\frac{dr}{dx} = \frac{13}{14}\frac{1}{Pr} \qquad \text{(f)}$$

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Separate variables and integrate

$$r = \left(\frac{\delta_t}{\delta}\right)^3 = C\,(x)^{-3/4} + \frac{13}{14}\frac{1}{Pr} \qquad \text{(g)}$$

C = constant. Use boundary condition on δ_t

$$\delta_t(x_o) = 0 \qquad \text{(h)}$$

Apply (h) to (g)

$$C = -\frac{13}{14}\frac{1}{Pr}x_o^{3/4} \qquad \text{(i)}$$

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(i) into (h)

$$\frac{\delta_t}{\delta} = \left\{\frac{13}{14}\frac{1}{Pr}\left[1 - \left(\frac{x_o}{x}\right)^{3/4}\right]\right\}^{1/3} \qquad \text{(5.16)}$$

Use (c) to eliminate δ in (5.16)

$$\delta_t = \left\{\frac{13}{14}\frac{1}{Pr}\left[1 - \left(\frac{x_o}{x}\right)^{3/4}\right]\right\}^{1/3} \sqrt{\frac{280}{13}}\sqrt{\frac{\nu x}{V_\infty}} \qquad \text{(5.17a)}$$

or

$$\frac{\delta_t}{x} = \frac{4.528}{Pr^{1/3}Re_x^{1/2}}\left\{\left[1 - \left(\frac{x_o}{x}\right)^{3/4}\right]\right\}^{1/3} \qquad \text{(5.17b)}$$

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Re_x is the local Reynolds

$$Re_x = \frac{V_\infty x}{\nu} \quad (5.18)$$

(ii) Nusselt Number

Local Nusselt number:

$$Nu_x = \frac{hx}{k} \quad (j)$$

h is given by

$$h = \frac{-k \dfrac{\partial T(x,0)}{\partial y}}{T_s - T_\infty} \quad (k)$$

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Use temperature solution (5.13) in (k)

$$h(x) = \frac{3}{2}\frac{k}{\delta_t} \quad (5.19)$$

Use (5.17a) to eliminate δ_t in (5.19)

$$h(x) = 0.331\ \frac{k}{x}\left\{1 - \left(\frac{x_o}{x}\right)^{3/4}\right\}^{-1/3} Pr^{1/3} Re_x^{1/2} \quad (5.20)$$

Substitute into (j)

$$Nu_x = 0.331\left\{1 - \left(\frac{x_o}{x}\right)^{3/4}\right\}^{-1/3} Pr^{1/3} Re_x^{1/2} \quad (5.21)$$

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(iii) Special Case: Plate with no Insulated Section

set $x_o = 0$ in above solution

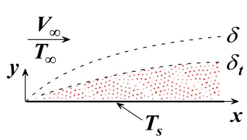

Fig. 5.9

$$\frac{\delta_t}{\delta} = \left\{\frac{13}{14}\frac{1}{Pr}\right\}^{1/3} = \frac{0.975}{Pr^{1/3}} \quad (5.22)$$

$$\frac{\delta_t}{x} = \frac{4.528}{Pr^{1/3} Re_x^{1/2}} \quad (5.23)$$

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$$h(x) = 0.331\frac{k}{x} Pr^{1/3} Re_x^{1/2} \quad (5.24)$$

$$Nu_x = 0.331 Pr^{1/3} Re_x^{1/2} \quad (5.25)$$

Accuracy of integral solution:

(1) for $Pr = 1$, $\delta_t / \delta = 1$. Set $Pr = 1$ in (5.22)

$$\frac{\delta_t}{\delta} = 0.975$$

Error is 2.5%

(2) Compare with Pohlhausen's solution. For $Pr > 10$

$$Nu_x = 0.339\, Pr^{1/3} \sqrt{Re_x}, \quad \text{for} \quad Pr > 10 \quad (4.72c)$$

Error is 2.4%

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Example 5.1:
Laminar Boundary Layer Flow over a Flat Plate: Uniform Surface Temperature

Use a linear profiles. Velocity:

$$u = a_0 + a_1 y \quad (a)$$

Boundary conditions

$$(1)\ u(x,0) = 0, \qquad (2)\ u(x,\delta) \approx V_\infty$$

$$u = V_\infty \frac{y}{\delta} \quad (b)$$

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Apply integral formulation of momentum, (5.5).

Result:

$$\frac{\delta}{x} = \sqrt{\frac{12}{Re_x}} \quad (5.26)$$

Temperature profile:

$$T = b_0 + b_1 y \quad (c)$$

Boundary conditions

$$(1)\ T(x,0) = T_s, \quad (2)\ T(x,\delta_t) \approx T_\infty$$

$$T = T_s + (T_\infty - T_s)\frac{y}{\delta_t} \quad (d)$$

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Apply integral formulation of energy, (5.7).

Result:

$$Nu_x = 0.289\, Pr^{1/3} \sqrt{Re_x} \left[1 - (x_o / x)^{3/4}\right]^{-1/3} \quad (5.27)$$

Special case: $x_o = 0$

$$Nu_x = 0.289\, Pr^{1/3} \sqrt{Re_x}$$

Comments

(i) Linear profiles give less accurate results than polynomials.

(ii) More accurate prediction of Nusselt number than viscous boundary layer thickness.

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5.7.3 Uniform Surface Flux

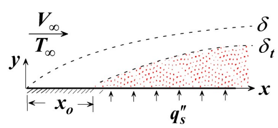

Fig. 5.10

- Flat plate
- Insulated section of length x_o
- Plate is heated with uniform flux q_s''
- Determine $h(x)$ and Nu_x

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$$q_s'' = h(x)[T_s(x) - T_\infty] \quad (a)$$

or

$$h(x) = \frac{q_s''}{T_s(x) - T_\infty}$$

Nusselt number:

$$Nu_x = \frac{q_s'' x}{k[T_s(x) - T_\infty]} \quad (b)$$

Apply conservation of energy to determine $T_s(x)$

$$-\alpha \frac{\partial T(x,0)}{\partial y} = \frac{d}{dx}\int_0^{\delta_t} u(T - T_\infty)dy \quad (5.7)$$

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Integral solution to $u(x, y)$:

$$\frac{u}{V_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \quad (5.9)$$

Assume temperature profile:

$$T = b_0 + b_1 y + b_2 y^2 + b_3 y^3 \quad (c)$$

Boundary conditions:

(1) $-k\dfrac{\partial T(x,0)}{\partial y} = q_s''$ (2) $T(x,\delta_t) \cong T_\infty$

(3) $\dfrac{\partial T(x,\delta_t)}{\partial y} \cong 0$ (4) $\dfrac{\partial^2 T(x,0)}{\partial y^2} = 0$

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$$T(x,y) = T_\infty + \left[\frac{2}{3}\delta_t - y + \frac{1}{3}\frac{y^3}{\delta_t^2}\right]\frac{q_s''}{k} \quad (5.29)$$

set $y = 0$ to obtain $T_s(x)$

$$T_s(x) = T(x,0) = T_\infty + \frac{2}{3}\frac{q_s''}{k}\delta_t \quad (5.30)$$

5.30 into (b)

$$Nu_x = \frac{3}{2}\frac{x}{\delta_t(x)} \quad (5.31)$$

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Must determine δ_t. Substitute (5.9) and (5.29) into (5.7)

$$\alpha = V_\infty \frac{d}{dx}\left\{\int_0^{\delta_t}\left[\frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\frac{y^3}{\delta^3}\right]\left[\frac{2}{3}\delta_t - y + \frac{1}{3}\frac{y^3}{\delta_t^2}\right]dy\right\} \quad (d)$$

Evaluating the integrals

$$\frac{\alpha}{V_\infty} = \frac{d}{dx}\left\{\delta_t^2\left[\frac{1}{10}\frac{\delta_t}{\delta} - \frac{1}{140}\left(\frac{\delta_t}{\delta}\right)^3\right]\right\} \quad (e)$$

For $Pr > 1$, $\delta_t / \delta < 1$

$$\frac{1}{140}\left(\frac{\delta_t}{\delta}\right)^3 << \frac{1}{10}\frac{\delta_t}{\delta} \quad (f)$$

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(f) into (e)

$$10\frac{\alpha}{V_\infty} = \frac{d}{dx}\left[\frac{\delta_t^3}{\delta}\right]$$

Integrate

$$10\frac{\alpha}{V_\infty}x = \frac{\delta_t^3}{\delta} + C \qquad \text{(g)}$$

Boundary condition:

$$\delta_t(x_o) = 0 \qquad \text{(h)}$$

Apply (h) to (g)

$$C = 10\frac{\alpha}{V_\infty}x_o \qquad \text{(i)}$$

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(i) into (g)

$$\delta_t = \left[10\frac{\alpha}{V_\infty}(x - x_o)\delta\right]^{1/3} \qquad \text{(j)}$$

Use(5.10) to eliminate δ in (j)

$$\delta_t = \left[10\frac{\alpha}{V_\infty}(x - x_o)\sqrt{\frac{280}{13}}\sqrt{\frac{280/13}{Re_x}}\,x\right]^{1/3}$$

or

$$\frac{\delta_t}{x} = \frac{3.594}{Pr^{1/3}Re_x}\left[1 - \frac{x_o}{x}\right]^{1/3} \qquad \text{(5.32)}$$

Surface temperature: (5.32) into (5.30)

$$T_s(x) = T_\infty + 2.396\frac{q_s''}{k}\left[1 - \frac{x_o}{x}\right]^{1/3}\frac{x}{Pr^{1/3}Re_x^{1/2}} \qquad \text{(5.33)}$$

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Nusselt number: (5.32) into (5.31)

$$Nu_x = 0.417\left[1 - \frac{x_o}{x}\right]^{-1/3}Pr^{1/3}Re_x^{1/2} \qquad \text{(5.34)}$$

Special case: $x_o = 0$

$$T_s(x) = T_\infty + 2.396\frac{q_s''}{k}\frac{x}{Pr^{1/3}Re_x^{1/2}} \qquad \text{(5.35)}$$

Does $T_s(x)$ increase or decrease with distance x?

Nusselt number:

$$Nu_x = 0.417\,Pr^{1/3}Re_x^{1/2} \qquad \text{(5.36)}$$

Exact solution:

$$Nu_x = 0.453\,Pr^{1/3}Re_x^{1/2} \qquad \text{(5.37)}$$

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Example 5.2: Laminar Boundary Layer Flow over a Flat Plate: Variable Surface Temperature

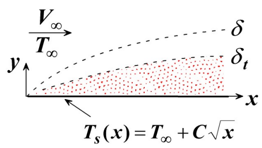

Fig. 5.11

- **Specified surface temperature**

$$T_s(x) = T_\infty + C\sqrt{x}$$

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- **Determine the local Nusselt number**

(1) Observations

- **Determine** $u(x, y)$ **and** $T(x, y)$
- **Variable** $T_s(x)$
- **Constant properties:** $T(x, y)$ **is independent of** $u(x, y)$

(2) Problem Definition. **Determine** $u(x,y)$ **and** $T(x, y)$

(3) Solution Plan

- **Start with the definition of** Nu_x
- **Apply the integral method to determine** $T(x, y)$

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(4) Plan Execution

(i) Assumptions

(1) Steady state
(2) Constant properties
(3) Two-dimensional
(4) Laminar flow (Re_x < 5×105)
(5) Viscous boundary layer flow (Re_x > 100)
(6) Thermal boundary layer (Pe > 100)
(7) Uniform upstream velocity and temperature
(8) Flat plate (9)
(9) Negligible changes in kinetic and potential energy

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(10) Negligible axial conduction

(11) Negligible dissipation

(12) No buoyancy ($b = 0$ or $g = 0$)

(ii) Analysis

$$Nu_x = \frac{hx}{k} \qquad \text{(a)}$$

(1.10) gives h

$$h = \frac{-k\dfrac{\partial T(x,0)}{\partial y}}{T_s(x) - T_\infty} \qquad (1.10)$$

To determine $T(x,y)$ use (5.7)

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$$-\alpha\frac{\partial T(x,0)}{\partial y} = \frac{d}{dx}\int_0^{\delta_t(x)} u(T - T_\infty)dy \qquad (5.7)$$

(5.9) gives $u(x,y)$

$$\frac{u}{V_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \qquad (5.9)$$

where

$$\delta = \frac{\sqrt{280/13}}{\sqrt{Re_x}}x = \sqrt{\frac{280}{13}\frac{x\nu}{V_\infty}} \qquad (5.10)$$

Assume

$$T(x,y) = b_0(x) + b_1(x)y + b_2(x)y^2 + b_3(x)y^3 \qquad \text{(a)}$$

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Boundary conditions:

(1) $T(x,0) = T_s(x)$

(2) $T(x,\delta_t) \cong T_\infty$

(3) $\dfrac{\partial T(x,\delta_t)}{\partial y} \cong 0$

(4) $\dfrac{\partial^2 T(x,0)}{\partial y^2} = 0$

$$T(x,y) = T_s(x) + [T_\infty - T_s(x)]\left[\frac{3}{2}\frac{y}{\delta_t} - \frac{1}{2}\frac{y^3}{\delta_t^3}\right] \qquad \text{(b)}$$

(b) into (1.10)

$$h(x) = \frac{3}{2}\frac{k}{\delta_t} \qquad \text{(c)}$$

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(c) into (a)

$$Nu_x = \frac{3}{2}\frac{x}{\delta_t} \quad \text{(d)}$$

- **Use (5.7) to determine δ_t**
- **(5.9) and (b) into (5.7), evaluating the integral**

$$\frac{3}{2}\alpha\frac{T_s(x)-T_\infty}{\delta_t} = \frac{d}{dx}\left\{[T_s(x)-T_\infty]V_\infty\delta\left[\frac{3}{20}\left(\frac{\delta_t}{\delta}\right)^2-\frac{3}{280}\left(\frac{\delta_t}{\delta}\right)^4\right]\right\} \quad \text{(e)}$$

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For $Pr > 1$

$$\frac{\delta_t}{\delta} < 1, \quad \text{for } Pr > 1 \quad \text{(5.15)}$$

Thus

$$\frac{3}{280}\left(\frac{\delta_t}{\delta}\right)^4 << \frac{3}{20}\left(\frac{\delta_t}{\delta}\right)^2$$

Simplify (e)

$$10\frac{\alpha}{\delta_t}[T_s(x)-T_\infty] = V_\infty\frac{d}{dx}\left[[T_s(x)-T_\infty]\delta\left(\frac{\delta_t}{\delta}\right)^2\right] \quad \text{(f)}$$

However

$$T_s(x)-T_\infty = C\sqrt{x} \quad \text{(g)}$$

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(5.10) and (g) into (f)

$$10\frac{\alpha}{\delta_t}\left[C\sqrt{x}\right] = V_\infty\frac{d}{dx}\left[C\sqrt{x}\sqrt{\frac{13}{280}\frac{V_\infty}{\nu x}}\,\delta_t^2\right]$$

Simplify

$$5\sqrt{\frac{280}{13}\frac{\alpha}{\nu}}\,[\nu/V_\infty]^{3/2}\sqrt{x}\,dx = \delta_t^2\,d\delta_t \quad \text{(h)}$$

Boundary condition on δ_t :

$$\delta_t(0) = 0 \quad \text{(i)}$$

Integrate (h) using (i)

$$\delta_t = \left[10\sqrt{280/13}\right]^{1/3}(Pr)^{-1/3}(\nu x/V_\infty)^{1/2} \quad \text{(j)}$$

(j) into (d)

$$Nu_x = 0.417\,Pr^{1/3}Re_x^{1/2} \quad \text{(5.38)}$$

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(3) Checking

- ***Dimensional check***
- ***Boundary conditions check***

(4) Comments

(i) (5.38) is identical with (5.36) for uniform flux

$$T_s(x) = T_\infty + 2.396 \frac{q_s''}{k} \frac{x}{Pr^{1/3} Re_x^{1/2}} \tag{5.35}$$

Rewrite (5.35)

$$T_s(x) = T_\infty + C\sqrt{x}$$

(ii) Use same procedure for other specified $T_s(x)$

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