CHAPTER 6
HEAT TRANSFER IN CHANNEL FLOW

6.1 Introduction

- **Important factors:**

(1) Laminar vs. turbulent flow

transition Reynolds number Re_{D_t} is

$$Re_{D_t} = \frac{\bar{u}D}{\nu} \approx 2300 \tag{6.1}$$

where

$D =$ tube diameter

$\bar{u} =$ mean velocity

$\nu =$ kinematic viscosity

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(2) Entrance vs. fully developed region

Based on velocity and temperature distribution: two regions:

(i) Entrance region

(ii) Fully developed region

(3) Surface boundary conditions

Two common thermal boundary conditions:

(i) Uniform surface temperature

(ii) Uniform surface heat flux

(4) Objective

Depends on the thermal boundary condition:

(i) Uniform surface temperature. Determine: axial variation of

(1) Mean fluid temperature

(2) Heat transfer coefficient

(3) Surface heat flux

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(ii) Uniform surface flux. Determine axial variation of:

(1) Mean fluid temperature

(2) Heat transfer coefficient

(3) Surface temperature

6.2 Hydrodynamic and Thermal Regions: General Features

- Uniform inlet velocity V_i and temperature T_i
- Developing boundary velocity and thermal boundary layers
- Two regions:

(1) Entrance region

(2) Fully developed region

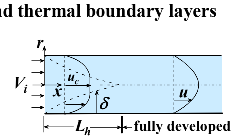

Fig. 6.1

6.2.1 Velocity Field

(1) Entrance Region (Developing Flow, $0 \le x \le L_h$)

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- Hydrodynamic entrance region
- Length L_h: hydrodynamic entrance length
- Streamlines are not parallel $(v_r \neq 0)$
- Core velocity $u_c = u_c(x)$
  (increasing or decreasing with x?)
- Pressure $p = p(x)$, $(dp/dx < 0)$
- $\delta < D/2$

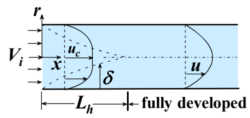

Fig. 6.1

(2) Fully Developed Flow Region

- $x > L_h$: fully developed flow
- Streamlines are parallel $(v_r = 0)$
- For 2-D, constant ρ : $\partial u / \partial x = 0$

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6.2.2 Temperature Field

- **Entrance Region (Developing Temperature, $0 \leq x \leq L_t$):**
- Thermal entrance region
- Length L_t: Thermal entrance length
- Core temperature T_c is uniform $(T_c = T_i)$
- $\delta_t < D/2$

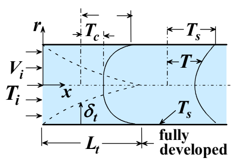

Fig. 6.2

(2) Fully Developed Temperature Region

- $x \geq L_t$ fully developed temperature
- $T = T(r, x)$ or $\partial T / \partial x \neq 0$
- Dimensionless temperature ϕ is invariant with x $(\partial \phi / \partial x = 0)$

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6.3 Hydrodynamic and Thermal Entrance Lengths

L_h and L_t are determined by:

(1) Scale analysis

(2) Analytic or numerical methods

6.3.1 Scale Analysis

(1) Hydrodynamic Entrance Length L_h

- Scaling of external flow:

$$\frac{\delta}{x} \sim \frac{1}{\sqrt{Re_x}} \qquad (4.16)$$

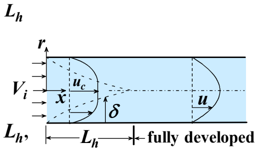

Fig. 6.1

Apply (4.16) to flow in tube: at $x = L_h$, $\delta \sim D$

$$\frac{D}{L_h} \sim \frac{1}{\sqrt{Re_{L_h}}} \qquad (a)$$

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Express Re_{L_h} in terms Re_D

$$Re_{L_h} = \frac{\bar{u} L_h}{\nu} = \frac{\bar{u} D}{\nu} \frac{L_h}{D} = Re_D \frac{L_h}{D}$$

(b) into (a)

$$\left(\frac{L_h / D}{Re_D} \right)^{1/2} \sim 1 \qquad (6.2)$$

(2) Thermal Entrance Length L_t

Scale for u: $u \sim \bar{u}$ (all Prandtl numbers)
Scale of δ_t: For external flow

$$\delta_t \sim L Re_L^{-1/2} Pr^{-1/2} \qquad (4.24)$$

Apply (4.24) for flow in tube: $L = L_t$, $\delta_t \sim D$

$$D \sim L_t Re_t^{-1/2} Pr^{-1/2} \qquad (a)$$

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Express Re_{L_t} in terms Re_D

$$Re_{L_t} = \frac{\bar{u} L_t}{\nu} = \frac{\bar{u} D}{\nu} \frac{L_t}{D} = Re_D \frac{L_t}{D} \qquad (b)$$

(b) into (a)

$$\left(\frac{L_t / D}{Re_D Pr} \right)^{1/2} \sim 1 \qquad (6.3)$$

$$\frac{L_t}{L_h} \sim Pr \qquad (6.4)$$

6.3.2 Analytic/Numerical Solutions: Laminar Flow

(1) Hydrodynamic Entrance Length L_h

$$\frac{L_h}{D_e} = C_h Re_D \qquad (6.5)$$

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D_e = equivalent diameter

$$D_e = \frac{4 A_f}{P}$$

A_f = flow area

P = perimeter

C_h = coefficient Table 6.1

Compare with scaling:

$$\left(\frac{L_h / D}{Re_D} \right)^{1/2} \sim 1 \qquad (6.2)$$

Rewrite (6.5)

$$\left(\frac{L_h / D}{Re_D} \right)^{1/2} = (C_h)^{1/2} \qquad (a)$$

Table 6.1
Entrance length coefficients C_h and C_t [1]

Geometry	C_h	C_t Uniform surface flux	C_t Uniform surface temperature
(circle)	0.056	0.043	0.033
a, b (rectangle) $a/b = 1$	0.090	0.066	0.041
a, b (rectangle) 2	0.085	0.057	0.049
a, b (rectangle) 4	0.075	0.042	0.054
(parallel plates)	0.011	0.012	0.008

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Example: Rectangular channel, $a/b = 2$,, Table 6.1 gives $C_h = 0.085$
Substitute into (a)

$$\left(\frac{L_h/D}{Re_D}\right)^{1/2} = (0.085)^{1/2} = 0.29 \qquad \text{(b)}$$

(2) Thermal Entrance Length L_t

$$\frac{L_t}{D_e} = C_t Pr Re_D \qquad (6.6)$$

C_t is given in Table 6.1
Compare with scaling

$$\left(\frac{L_t/D}{Re_D Pr}\right)^{1/2} \sim 1 \qquad (6.3)$$

Rewrite (6.6):

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$$\left(\frac{L_t/D}{Re_D Pr}\right)^{1/2} = C_t^{1/2} \qquad \text{(b)}$$

Example: Rectangular channel, $a/b = 2$,, Table 6.1 gives $C_t = 0.049$ gives

$$\left(\frac{L_t/D}{Re_D Pr}\right)^{1/2} = (0.049)^{1/2} = 0.22 \qquad \text{(c)}$$

Turbulent flow: Experimental results:
- L_h and L_t are shorter than in laminar flow

Rule of thumb :

$$10 < \frac{L_n}{D} < 60 \qquad (6.7a)$$

$$40 < \frac{L_n}{D} < 100 \qquad (6.7b)$$

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6.4 Channels with Uniform Surface Heat Flux q_s''

When surface heat flux is uniform
Surface temperature is variable

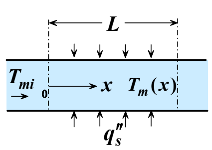

Fig. 6.3

- Section length L
- Inlet temperature: $T_{mi} = T_m(0)$
- Surface flux q_s''

Determine:

(1) Total heat transfer
(2) Mean temperature variation $T_m(x)$
(3) Surface temperature variation $T_s(x)$

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Total heat:

$$q_s = q_s'' A_s = q_s'' P x \qquad (6.8)$$

A_s = surface area
P = perimeter

Conservation of energy:

Assumptions:

(1) Steady state
(2) No energy generation
(3) Negligible changes in kinetic and potential energy
(4) No axial conduction

$$q_s = q_s'' P x = \dot{m} c_p [T_m(x) - T_{mi}]$$

or

$$T_m(x) = T_{mi} + \frac{q_s'' P}{\dot{m} c_p} x \qquad (6.9)$$

$\dot{m}$ = mass flow rate

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c_p = specific heat

(6.9) applies to any region and any flow (laminar, turbulent or mixed)

Use heat transfer analysis to determine surface temperature $T_s(x)$

Newton's law of cooling

$$q_s'' = h(x)[T_s(x) - T_m(x)]$$

Solve for $T_s(x)$

$$T_s(x) = T_m(x) + \frac{q_s''}{h(x)}$$

Use (6.9) to eliminate $T_m(x)$

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$$T_s(x) = T_{mi} + q_s'' \left[\frac{Px}{\dot{m} c_p} + \frac{1}{h(x)} \right] \qquad (6.10)$$

$h(x)$ is needed in (6.10) to determine $T_s(x)$

To determine $h(x)$:
(1) Laminar or turbulent flow?
(2) Entrance or fully developed region?

Example 6.2: Maximum Surface Temperature

- Water flows through tube
- Mean velocity = 0.2 m/s
- $T_{mi} = 20^\circ$C

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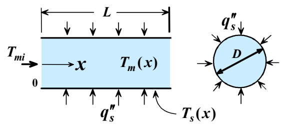
- $T_{mo} = 80^{o}C$
- $D = 0.5$ cm
- Uniform surface heat flux = 0.6 W/cm^2
- Fully developed flow at outlet
- Nusselt number for laminar fully developed flow



$$Nu_D = \frac{hD}{k} = 4.364 \quad \text{(A)}$$

Determine the maximum surface temperature

(1) Observations

- Uniform surface flux
- $T_s = T_s(x)$, maximum at the outlet

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- Laminar or turbulent flow? Check Re_D
- Is outlet fully developed? Check L_h and L_t
- Uniform Nusselt number (h is constant)
- Length of tube section is unknown

(2) Problem Definition

(i) Determine L
(ii) Determine $T_s(L)$

(3) Solution Plan

(i) Apply conservation of energy
(ii) Compute Re_D
(iii) Calculate L_h and L_t
(iv) Apply uniform flux analysis
(v) If applicable use (A) to determine h

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(4) Plan Execution

(i) Assumptions

- Steady state
- Constant properties
- Axisymmetric flow
- Uniform surface heat flux
- Negligible changes in kinetic and potential energy
- Negligible axial conduction
- Negligible dissipation

(ii) Analysis

Conservation of energy:

$$\pi D L q_s'' = \dot{m} c_p (T_{mo} - T_{mi}) \quad \text{(a)}$$

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c_p = specific heat, J/kg-°C
D = tube diameter = 0.5 cm = 0.005 m
L = tube length, m
$\dot{m}$ = mass flow rate, kg/s
T_{mi} = mean temperature at the inlet = 20°C
T_{mo} = mean temperature at the outlet = 80°C
q_s'' = surface heat flux = 0.6 W/cm² = 6000 W/m²

From (a)

$$L = \frac{\dot{m} c_p (T_{mo} - T_{mi})}{\pi D q_s''} \quad \text{(b)}$$

Conservation of mass:

$$\dot{m} = (\pi / 4) D^2 \rho \bar{u} \quad \text{(c)}$$

where

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$\bar{u}$ = mean flow velocity = 0.2 m/s
ρ = density, kg/m³

Surface temperature: Apply (6.10)

$$T_s(x) = T_{mi} + q_s'' \left[\frac{Px}{\dot{m} c_p} + \frac{1}{h(x)} \right] \quad (6.10)$$

h = local heat transfer coefficient, W/m²-°C
P = tube perimeter, m
$T_s(x)$ = local surface temperature, °C
x = distance from inlet of heated section, m

Perimeter P:

$$P = \pi D \quad \text{(d)}$$

Maximum surface temperature: set $x = L$ in (6.10)

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$$T_s(L) = T_{mi} + q_s'' \left[\frac{PL}{\dot{m} c_p} + \frac{1}{h(L)} \right] \quad \text{(e)}$$

Determine $h(L)$: Is flow laminar or turbulent? Compute Re_D

$$Re_D = \frac{\bar{u} D}{\nu} \quad \text{(f)}$$

Properties $\bar{T}_m$

$$\bar{T}_m = \frac{T_{mi} + T_{mo}}{2} \quad \text{(g)}$$

$$\bar{T} = \frac{(20 + 80)(^\circ\text{C})}{2} = 50^\circ\text{C}$$

For water:

c_p = 4182 J/kg-°C
k = 0.6405 W/m-°C

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$Pr = 3.57$

$\nu = 0.5537 \times 10^{-6}\ \text{m}^2/\text{s}$

$\rho = 988\ \text{kg/m}^3$

Use (g)

$$Re_D = \frac{0.2(\text{m/s})0.005(\text{m})}{0.5537 \times 10^{-6}(\text{m}^2/\text{s})} = 1806,\ \text{laminar flow}$$

Compute L_h and L_t using (6.5) and (6.6)

$$\frac{L_h}{D_e} = C_h Re_D \qquad (6.5)$$

$$\frac{L_t}{D_e} = C_t Pr Re_D \qquad (6.6)$$

$C_h = 0.056$ (Table 6.1)

$C_t = 0.043$ (Table 6.1)

$L_h = 0.056 \times 0.005\ (\text{m}) \times 1806 = 0.506\ \text{m}$

$L_t = 0.043 \times 0.005\ (\text{m}) \times 1806 \times 3.57 = 1.386\ \text{m}$

Is L smaller or larger than L_h and L_t ?

Compute L using (b). Use (c) to compute $\dot{m}$

$\dot{m} = 988(\text{kg/m}^3)\ 0.2(\text{m/s})\pi(0.005)^2(\text{m}^2)/4 = 0.00388\text{kg/s}$

Substitute into (b)

$$L = \frac{0.00388(\text{kg/s})\,4182(\text{J/kg}-^{\circ}\text{C})(80-20)(^{\circ}\text{C})}{0.005(\text{m})\,0.6\,(\text{W/cm}^2)10^4(\text{cm}^2/\text{m}^2)} = 10.33\ \text{m}$$

L is larger than both L_h and L_t .
Flow is fully developed at the outlet

Equation (A) is applicable

$$Nu_D = \frac{hD}{k} = 4.364 \qquad (A)$$

(iii) Computations. Apply (A)

$$h(L) = 4.364\ \frac{0.6405(\text{W/m}-^{\circ}\text{C})}{0.005(\text{m})} = 559\ \text{W/m}^2\text{-}^{\circ}\text{C}$$

Use (e)

$$T_s(L) = 20^{\circ}\text{C} + 6000(\text{W/m}^2)\left[\frac{\pi 0.005(\text{m})0.43(\text{m})}{0.00388(\text{kg/s})4182(\text{J/kg}-^{\circ}\text{C})} + \frac{1}{559(\text{W/m}^2-^{\circ}\text{C})}\right]$$

$$T_s(L) = 90.7^{\circ}\text{C}$$

(iv) Checking. *Dimensional check*:

Quantitative checks: (1) Alternate approach: apply Newton's law at outlet

$$q_s'' = h(L)[T_s(L) - T_{mo}] \quad \text{(i)}$$

solve for $T_s(L)$

$$T_s(L) = T_{mo} + \frac{q_s''}{h} = 80\ (^oC) + \frac{0.6(W/cm^2)\times 10^4(cm^2/m^2)}{559(W/m^2 - {}^oC)} = 90.7^oC$$

(2) Compare value of h with Table 1.1

Limiting check: For $T_{mi} = T_{mo}$, $L = 0$. Set $T_{mi} = T_{mo}$ in (b) gives $L = 0$.

(3) Comments.

- In laminar flow local h depends on local flow condition: entrance vs. fully developed
- Check Re_D to determine:

(i) If flow is laminar or turbulent
(ii) Entrance or fully developed

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6.5 Channels with Uniform Surface Temperature

When surface temperature is uniform, surface heat flux is variable

- Surface temperature: T_s
- Inlet temperature: $T_{mi} = T_m(0)$
- Section length: L

Determine

(1) Mean temperature variation $T_m(x)$

(2) Total heat q_s

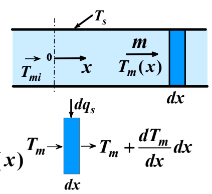

Fig. 6.4

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(3) Surface flux variation $q_s''(x)$

Analysis

Apply conservation of energy to element dx

Assumptions

(1) Steady state

(2) No energy generation

(3) Negligible changes in kinetic and potential energy

(4) No axial conduction

$$dq_s = m\, c_p dT_m \quad \text{(a)}$$

Newton's law:

$$dq_s = h(x)[T_s - T_m(x)]P dx$$

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Combine (a) and (b)

$$\frac{dT_m}{T_s - T_m(x)} = \frac{P}{\dot{m}\, c_p} h(x)dx \qquad \text{(c)}$$

Integrate from $x = 0$ $(T_m = T_m(0) = T_{mi})$ to x $(T_m = T_m(x))$

$$\ln\left[\frac{T_m(x) - T_s}{T_{mi} - T_s}\right] = -\frac{P}{\dot{m}\, c_p}\int_0^x h(x)dx$$

Must determine $h(x)$. Introduce $\bar{h}$

$$\bar{h} = \frac{1}{x}\int_0^x h(x)dx \qquad (6.12)$$

(6.12) into (6.11)

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$$T_m(x) = T_s + (T_{mi} - T_s)\exp[-\frac{P\bar{h}}{\dot{m}\, c_p}x] \qquad (6.13)$$

(6.13) applies to any region and any flow (laminar, turbulent or mixed)

To determine $h(x)$:
(1) Is flow laminar or turbulent flow?
(2) Entrance or fully developed region?

Total heat: Apply conservation of energy:

$$q_s = \dot{m}\, c_p[T_m(x) - T_{mi}]$$

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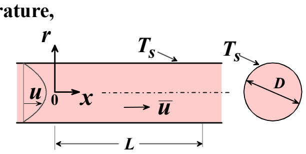
Surface flux: Apply Newton's law:

$$q_s''(x) = h(x)[T_s - T_m(x)] \qquad (6.15)$$

Properties: At mean of inlet and outlet temperatures

Example 6.3: Required Tube Length

- Air flows through tube
- Uniform surface temperature, T_s = 130°C
- Mean velocity = 2 m/s,
- T_{mi} = 35°C
- D = 1.0 cm



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- Nusselt number for laminar fully developed flow

$$Nu_D = \frac{hD}{k} = 3.657 \quad \text{(A)}$$

Determine: tube length to raise temperature to $T_{mo} = 105^\circ\text{C}$

(1) Observations

- Laminar or turbulent flow? Check Re_D
- Uniform surface temperature
- Uniform Nusselt number (h is constant) for fully developed laminar flow
- Length of tube is unknown

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(2) Problem Definition. Determine tube length needed to raise temperature to specified level

(3) Solution Plan.

- Use uniform surface temperature analysis
- Compute Re_D. Laminar or turbulent?

(4) Plan Execution

(i) Assumptions

- Steady state
- Fully developed flow
- Constant properties

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- Uniform surface temperature
- Negligible changes in kinetic and potential energy
- Negligible axial conduction
- Negligible dissipation

(ii) Analysis

$$T_m(x) = T_s + (T_{mi} - T_s)\exp\left[-\frac{P\bar{h}}{\dot{m}\, c_p}x\right] \quad (6.13)$$

c_p = specific heat, J/kg $-^\circ$C

$\bar{h}$ = average h, W/m$^2-^\circ$C

$\dot{m}$ = flow rate, kg/s

P = perimeter, m

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$T_m(x)$ = mean temperature at x, ^oC

T_{mi} = mean inlet temperature = 35 ^oC

T_s = surface temperature = 130 ^oC

x = distance from inlet, m

Apply (a) at the outlet ($x = L$), solve for L

$$L = \frac{m\, c_p}{P\bar{h}} \ln \frac{T_s - T_{mi}}{T_s - T_{mo}} \quad \text{(a)}$$

T_{mo} = outlet temperature = 105 ^oC

Properties: at $\bar{T}_m$

$$\bar{T}_m = \frac{T_{mi} + T_{mo}}{2} \quad \text{(b)}$$

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$$P = \pi D \quad \text{(c)}$$

$$m = \pi \frac{D^2}{4} \rho \bar{u} \quad \text{(d)}$$

D = inside tube diameter = 1 cm = 0.01 m

$\bar{u}$ = mean flow velocity = 2 m/s

ρ = density, kg/m^3

For fully developed laminar flow

$$Nu_D = \frac{hD}{k} = 3.657 \quad \text{(e)}$$

h = heat transfer coefficient, W/m^2 – ^oC

k = thermal conductivity of air, W/m – ^oC

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$$h = \bar{h} = 3.657 \frac{k}{D}, \text{ for laminar fully developed} \quad \text{(f)}$$

Compute: Reynolds number

$$Re_D = \frac{\bar{u} D}{\nu} \quad \text{(g)}$$

Use (b)

$$\bar{T}_m = \frac{(35 + 105)(^oC)}{2} = 70^oC$$

Properties:

c_p = 1008.7 J/kg – ^oC

$k = 0.02922$ W/m – ^oC

$Pr = 0.707$

$\nu = 19.9 \times 10^{-6}$ m^2/s

$\rho = 11.0287$ kg/m^3

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Use (f)

$$Re_D = \frac{2(\text{m/s})0.01(\text{m})}{19.9\times10^{-6}(\text{m}^2/\text{s})} = 1005\text{, flow is laminar}$$

(iii) Computations

$P = \pi\, 0.01(\text{m}) = 0.03142$ m

$$m = \pi\frac{(0.01)^2(\text{m}^2)}{4}1.0287(\text{kg/m}^3)2(\text{m/s}) = 0.0001616 \text{ kg/s}$$

$$\bar{h} = 3.657\ \frac{0.02922(\text{W/m}-^\circ\text{C})}{0.01(\text{m})} = 10.69 \quad \text{W/m}^2-^\circ\text{C}$$

Substitute into (a)

$$L = \frac{0.0001616(\text{kg/s})1008.7(\text{J/kg}-^\circ\text{C})}{0.03142(\text{m})10.69(\text{W/m}^2-^\circ\text{C})}\ln\frac{(130-35)(^\circ\text{C})}{(130-105)(^\circ\text{C})} = 0.65 \text{ m}$$

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(iv) Checking. *Dimensional check*

(i) $L = 0$ for $T_{mo} = T_{mi}$. Set $T_{mo} = T_{mi}$ in (a) gives $L = 0$

(ii) $L = \infty$ for $T_{mo} = T_s$. Set $T_{mo} = T_s$ in (a) gives $L = \infty$

Quantitative checks: (i) Approximate check:

Energy added at the surface = Energy gained by air (h)

Energy added at surface $\approx \bar{h}\pi DL(T_s - \bar{T}_m)$ (i)

Energy gained by air = $mc_p(T_{mo} - T_{mi})$ (j)

(j) and (k) into (i), solve for L

$$L = \frac{m\,c_p(T_{mo} - T_{mi})}{\bar{h}\,\pi D(T_s - \bar{T}_m)} \quad \text{(k)}$$

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$$L = \frac{0.0001616(\text{kg/s})1008.7(\text{J/kg}-^\circ\text{C})(105-35)(^\circ\text{C})}{10.69(\text{W/m}^2-^\circ\text{C})\pi(0.01)(\text{m})(130-70)(^\circ\text{C})} = 0.57 \text{ m}$$

(ii) Value of h is low compared with Table 1.1. Review solution. Deviation from Table 1.1 are expected

(5) Comments. This problem is simplified by two conditions: fully developed and laminar flow

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6.6 Determination of Heat Transfer Coefficient $h(x)$ and Nusselt Number Nu_D

Two Methods:

(1) Scale analysis

(2) Analytic or numerical solutions

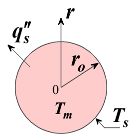

Fig. 6.5

6.6.1 Scale Analysis

Fourier's law and Newton's law

$$h = \frac{-k\dfrac{\partial T(r_o,x)}{\partial r}}{T_m - T_s} \qquad (6.16)$$

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Scales:

$$r \sim \delta_t \qquad (a)$$

$$\frac{\partial T(r_o,x)}{\partial r} \sim \frac{T_m - T_s}{\delta_t} \qquad (b)$$

(a) and (b) into (6.16)

$$h \sim \frac{k\dfrac{(T_m - T_s)}{\delta_t}}{T_m - T_s}$$

or

$$h \sim \frac{k}{\delta_t} \qquad (6.17)$$

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Nusselt number:

$$Nu_D = \frac{hD}{k}$$

(6.17) into the above

$$Nu_D \sim \frac{D}{\delta_t} \qquad (6.18)$$

Entrance region: $\delta_t < D$, $Nu_D > 1$

Special case: fully developed region

$$\delta_t(x) \sim D$$

(6.18) gives

$$Nu_D \sim 1 \text{ (fully developed)} \qquad (6.19)$$

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δ_t **in the entrance region: For all** Pr

$$\delta_t \sim x\,Pr^{-1/2} Re_x^{-1/2} \tag{4.24}$$

(4.24) into (6.18)

$$Nu_D \sim \frac{D}{x} Pr^{1/2} Re_x^{1/2} \tag{c}$$

Express in terms of Re_D

$$Re_x = \frac{\bar{u}x}{\nu} = \frac{\bar{u}D}{\nu}\frac{x}{D} = Re_D \frac{x}{D} \tag{d}$$

(d) into (c)

$$Nu_D \sim \left(\frac{D}{x}\right)^{1/2} Pr^{1/2} Re_x^{1/2} \tag{6.20a}$$

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or

$$\frac{Nu_D}{\left(\frac{PrRe}{x/D}\right)^{1/2}} \sim 1 \tag{6.20b}$$

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6.6.2 Basic Considerations for the Analytical Determination of Heat Flux, Heat Transfer Coefficient and Nusselt Number

(1) Fourier's law and Newton's law

$$q_s'' = -k\frac{\partial T(x, r_o)}{\partial r} \tag{a}$$

Define dimensionless variables

$$\theta \equiv \frac{T - T_s}{T_i - T_s}, \quad \xi = \frac{x/D}{Re_D Pr}, \quad R = \frac{r}{r_o}$$

$$v_x^* = \frac{v_x}{\bar{u}}, \quad v_r^* = \frac{v_r}{\bar{u}}, \quad Re_D = \frac{\bar{u}D}{\nu} \tag{6.21}$$

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(6.21) into (a)

$$q''_s(\xi) = \frac{k}{r_o}(T_s - T_i)\frac{\partial\theta(\xi,1)}{\partial R} \quad (6.22)$$

Newton's law

$$h(\xi) = \frac{q''_s}{T_m - T_s} \quad (6.23)$$

Combine (6.22) and (6.23)

$$h(\xi) = \frac{k(T_s - T_i)}{r_o(T_m - T_s)}\frac{\partial\theta(\xi,1)}{\partial R} = -\frac{k}{r_o}\frac{1}{\theta_m(\xi)}\frac{\partial\theta(\xi,1)}{\partial R} \quad (6.24)$$

Dimensionless mean temperature θ_m:

$$\theta_m \equiv \frac{T_m - T_s}{T_i - T_s} \quad (6.25)$$

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Nusselt number:

$$Nu(\xi) = \frac{h(\xi)D}{k} = \frac{h(\xi)2r_o}{k} \quad (6.26)$$

(6.24) into (6.26)

$$Nu(\xi) = \frac{-2}{\theta_m(\xi)}\frac{\partial\theta(\xi,1)}{\partial R} \quad (6.27)$$

Determine: $q''_s(\xi)$, $h(\xi)$ and: $Nu(\xi)$

Find $\theta(\xi, R)$. Apply energy equation

(2) The Energy Equation

Assumptions

- Steady state

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- Laminar flow
- Axisymmetric
- Negligible gravity
- Negligible dissipation
- Negligible changes in kinetic and potential energy
- Constant properties

$$\rho c_p\left(v_r\frac{\partial T}{\partial r} + \frac{\partial T}{\partial\theta} + v_z\frac{\partial T}{\partial z}\right) = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right] \quad (2.24)$$

Replace z by x, express in dimensionless form

$$v_x^*\frac{\partial\theta}{\partial\xi} + 2Re_D Pr\, v_r^*\frac{\partial\theta}{\partial R} = \frac{4}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\theta}{\partial R}\right) + \frac{1}{(Re_D Pr)^2}\frac{\partial^2\theta}{\partial\xi^2} \quad (6.28)$$

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$$Pe = Re_D Pr\ , \quad \text{Peclet number} \tag{2.29}$$

- Third term: radial conduction
- Fourth term: axial conduction
- Neglect axial conduction for:

$$Pe = PrRe_D \geq 100 \tag{6.30}$$

Simplify (6.28)

$$v_x^* \frac{\partial \theta}{\partial \xi} + 2Re_D Pr\, v_r^* \frac{\partial \theta}{\partial R} = \frac{4}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \theta}{\partial R}\right) \tag{6.31}$$

(3) Mean (Bulk) Temperature T_m

Need a reference local temperature. Use $T_m(x)$

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$$mc_p T_m = \int_0^{r_o} \rho c_p v_x T 2\pi r dr \tag{a}$$

where

$$m = \int_0^{r_o} \rho v_x 2\pi r dr \tag{b}$$

(b) into (a), assume constant properties

$$T_m = \frac{\int_0^{r_o} v_x T r dr}{\int_0^{r_o} v_x r dr} \tag{6.32a}$$

In dimensionless form:

$$\theta_m = \frac{T_m - T_s}{T_i - T_s} = \frac{\int_0^1 v_x^* \theta R dR}{\int_0^1 v_x^* R dR} \tag{6.32b}$$

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6.7 Heat Transfer Coefficient in the Fully Developed Temperature Region

6.7.1 Definition of Fully Developed Temperature Profile

Fully developed temperature:

$$\phi = \frac{T_s(x) - T(r,x)}{T_s(x) - T_m(x)}$$

Let

$$x/d > 0.05 Re_D Pr \tag{6.33}$$

Definition:

For fully developed temperature ϕ is independent of x

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Therefore

$$\phi = \phi(r) \tag{6.34}$$

From (6.34)

$$\frac{\partial \phi}{\partial x} = 0 \tag{6.35}$$

(6.33) and (6.35):

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left[\frac{T_s(x) - T(r,x)}{T_s(x) - T_m(x)}\right] = 0 \tag{6.36a}$$

Expand and use (6.33)

$$\frac{dT_s}{dx} - \frac{\partial T}{\partial x} - \phi(r)\left[\frac{dT_s}{dx} - \frac{dT_m}{dx}\right] = 0 \tag{6.36b}$$

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6.7.2 Heat Transfer coefficient and Nusselt number

$$h = \frac{-k\dfrac{\partial T(r_o,x)}{\partial r}}{T_m - T_s} \tag{6.16}$$

Use(6.33) to form $\partial T(r_o,x)/\partial r$, substitute into (6.16)

$$h = -k\frac{d\phi(r_o)}{dr} \tag{6.37}$$

Conclusion:

The heat transfer coefficient in the fully developed region is constant regardless of boundary condition

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Nusselt number:

$$Nu_D = \frac{hD}{h} = -D\frac{d\phi(r_o)}{dr} \tag{6.38}$$

Entrance region scaling result:

$$Nu_D \sim 1 \quad \text{(fully developed)} \tag{6.19}$$

Scaling of fully developed region:

scale for $\partial T(r_o,x)/\partial r$

$$\frac{\partial T(r_o,x)}{\partial r} \sim \frac{T_s - T_m}{D}$$

Substitute into (6.16)

$$h \sim \frac{k}{D} \tag{6.39}$$

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(6.39) into (6.38)

$$Nu_D \sim 1 \quad \text{(fully developed)} \tag{6.40}$$

6.7.3 Fully Developed Region for Tubes at Uniform Surface flux

- Uniform flux
- Determine
  - $T_s(x)$
  - h

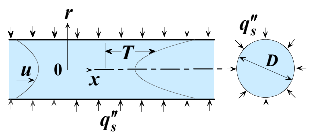

Fig. 6.6

Newton's law:

$$q_s'' = h[T_s(x) - T_m(x)] \tag{a}$$

$T_s(x)$ and $T_m(x)$ are unknown

q_s'' and h are constant

(a) gives:

$$[T_s(x) - T_m(x)] = \text{constant} \tag{b}$$

Differentiate (b)

$$\frac{dT_s}{dx} = \frac{dT_m}{dx} \tag{c}$$

(c) into (6.36b)

$$\frac{\partial T}{\partial x} = \frac{dT_s}{dx} \tag{d}$$

Combine (c) and (d)

$$\frac{\partial T}{\partial x} = \frac{dT_s}{dx} = \frac{dT_m}{dx} \quad \text{(for constant } q_s''\text{)} \tag{6.41}$$

To determine h form (6.16) :

Determine: $T(r,r)$, $T_m(x)$ and $T_s(x)$

Conservation of energy for dx

$$q_s'' P dx + \dot{m} c_p T_m = \dot{m} c_p \left[T_m + \frac{dT_m}{dx} dx \right]$$

Simplify

$$\frac{dT_m}{dx} = \frac{q_s'' P}{\dot{m} c_p} = \text{constant} \tag{6.42}$$

(6.42) into (6.41)

$$\frac{\partial T}{\partial x} = \frac{dT_s}{dx} = \frac{dT_m}{dx} = \frac{q_s'' P}{\dot{m} c_p} = \text{constant} \tag{6.43}$$

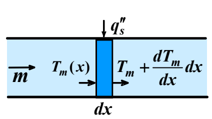

Fig. 6.7

Conclusion: $T(x,r)$, $T_m(x)$ and $T_s(x)$ are linear with x

Integrate(6.43)

$$T_m(x) = \frac{q_s'' P}{m c_p} x + C_1 \quad \text{(e)}$$

C_1 = constant

Boundary condition:

$$T_m(0) = T_{mi} \quad \text{(f)}$$

Apply (e) to (f)

$$C_1 = T_{mi}$$

(e) becomes

$$T_m(x) = T_{mi} + \frac{q_s'' P}{m c_p} x \quad \text{(6.44)}$$

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Determine $T(r,x)$ and $T_s(x)$

Apply energy equation (2.23) in the fully developed region

Assumptions

- Negligible axial conduction
- Negligible dissipation
- Fully developed, $v_r = 0$

$$\rho c_p v_x \frac{\partial T}{\partial x} = \frac{k}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) \quad \text{(6.45)}$$

$$v_x = 2\overline{u}\left[1 - \frac{r^2}{r_o^2} \right] \quad \text{(6.46)}$$

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However

$$m = \pi r_o^2 \rho \overline{u}$$

$$P = 2\pi r_o$$

equation (g) becomes

$$\frac{4q_s''}{r_o}\left[1 - \frac{r^2}{r_o^2} \right] = \frac{k}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r} \right) \quad \text{(6.47)}$$

Boundary conditions:

$$\frac{\partial T(0,x)}{\partial r} = 0 \quad \text{(6.48a)}$$

$$k \frac{\partial T(r_o,x)}{\partial r} = q_s'' \quad \text{(6.48b)}$$

Integrate (6.47)

$$\frac{4}{r_o} q_s'' \left[\frac{r^2}{2} - \frac{r^3}{4r_o^2} \right] = kr \frac{\partial T}{\partial r} + f(x) \quad \text{(h)}$$

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$f(x)$ = "constant" of integration

Boundary condition (6.48a)

$$f(x) = 0$$

(h) becomes

$$\frac{\partial T}{\partial r} = \frac{4q_s''}{kr_o}\left[\frac{r}{2} - \frac{r^3}{4r_o^2}\right]$$

Integrate

$$T(r,x) = \frac{4q_s''}{kr_o}\left[\frac{r^2}{4} - \frac{r^4}{4r_o^2}\right] + g(x) \qquad (6.49)$$

$g(x)$ = "constant" of integration

- Boundary condition (6.48b) is satisfied
- Use solution to $T_m(x)$ to determine $g(x)$

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Substitute(6.46) and (6.49) into (6.32a)

$$T_m(x) = \frac{7}{24}\frac{r_o q_s''}{k} + g(x) \qquad (6.50)$$

Two equations for $T_m(x)$: (6.44) and (6.50). Equating

$$g(x) = T_{mi} - \frac{7}{24}\frac{r_o q_s''}{k} + \frac{Pq_s''}{mc_p}x \qquad (6.51)$$

(6.51) into (6.49)

$$T(r,x) = T_{mi} + \frac{4q_s''}{kr_o}\left[\frac{r^2}{4} - \frac{r^3}{16r_o^2}\right] - \frac{7}{24}\frac{r_o q_s''}{k} + \frac{Pq_s''}{mc_p}x \qquad (6.52)$$

Surface temperature $T_s(x)$: set $r = r_o$ in (6.52)

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$$T_s(x) = T_{mi} + \frac{11}{24}\frac{r_o q_s''}{k} + \frac{Pq_s''}{mc_p}x \qquad (6.53)$$

$T(r,x)$, $T_m(x)$ and $T_s(x)$ are determined

(6.44), (6.52) and (6.53) into (6.33)

$$\phi(r) = 1 - \frac{24}{11}\frac{1}{r_o^2}\left[r^2 - \frac{r^4}{4r_o^2}\right] + \frac{24}{11}\frac{Pq_s''}{mc_p}x + \frac{7}{11}x \qquad (6.54)$$

Differentiate(6.54) and use (6.38)

$$Nu_D = \frac{48}{11} = 4.364\text{, laminar fully developed} \qquad (6.55)$$

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Comments:

- **(6.55) applies to:**
- **Laminar flow in tubes**
- **Fully developed velocity and temperature**
- **Uniform surface heat flux**
- **Nusselt number is independent of Reynolds and Prandtl numbers**
- **Scaling result:**

$$Nu_D \sim 1 \tag{6.40}$$

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6.7.4 Fully Developed Region for Tubes at Uniform Surface Temperature

- **Fully developed**
- **Uniform surface temperature** T_s

Determine: Nu_D **and** h

Assumptions:

- **Neglect axial conduction**
- **Neglect dissipation**
- **Fully developed:** $v_r = 0$

Energy equation (2.24):

$$\rho c_p v_x \frac{\partial T}{\partial x} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \tag{6.45}$$

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Boundary conditions:

$$\frac{\partial T(0,x)}{\partial r} = 0 \tag{6.56a}$$

$$T(r_o,x) = T_s \tag{6.56b}$$

Axial velocity

$$v_x = 2\bar{u}\left[1 - \frac{r^2}{r_o^2}\right] \tag{6.46}$$

Eliminate $\partial T/\partial x$ **in equation (6.45). Use (6.36a)**

$$\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x}\left[\frac{T_s(x) - T(r,x)}{T_s(x) - T_m(x)}\right] = 0 \tag{6.36a}$$

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for $T_s(x) = T_s$, (6.36a) gives

$$\frac{\partial T}{\partial x} = \frac{T_s - T(r,x)}{T_s - T_m(x)}\frac{dT_m}{dx} \qquad (6.57)$$

(6.46) and (6.57) into (6.45)

$$\rho c_p \overline{u}\left[1 - \frac{r^2}{r_o^2}\right]\frac{T_s - T(r,x)}{T_s - T_m(x)}\frac{dT_m}{dx} = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) \qquad (6.58)$$

Result: Solution to (6.58) by infinite power series:

$$Nu_D = 3.657 \qquad (6.59)$$

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6.7.5 Nusselt Number for Laminar Fully Developed Velocity and Temperature in Channels of Various Cross Sections

- Analytical and numerical solutions
- Results for two classes of boundary conditions:

  (1)Uniform surface flux

  (2)Uniform surface temperature
- Nusselt number is based on the equivalent diameter

$$D_e = \frac{4A_f}{P} \qquad (6.60)$$

- Results: Table 6.2

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Table 6.2
Nusselt number for laminar fully developed conditions in channels of various cross-sections [3]

Channel geometry	$\frac{a}{b}$	Nusselt number Nu_D Uniform surface flux	Nusselt number Nu_D Uniform surface temperature
		4.364	3.657
	1	3.608	2.976
	2	4.123	3.391
	4	5.331	4.439
	8	6.49	5.597
	∞	8.235	7.541
		3.102	2.46

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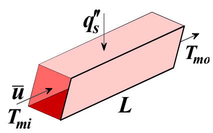
- Compare with scaling

$$Nu_D \sim 1 \quad \text{(fully developed)} \qquad (6.40)$$

Example 6.4: Maximum Surface Temperature in an Air Duct

- 4 cm × 4 cm square duct
- Uniform heat flux = 590 W/m^2
- Heating air from 40 °C to 120° C
- $\bar{u}$ = 0.32 m/s
- No entrance effects (fully developed)



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Determine: Maximum surface temperature

(1) Observations

- Uniform surface flux
- Variable Surface temperature, $T_s(x)$, maximum at outlet
- Compute the Reynolds number
- Velocity and temperature are fully developed
- The heat transfer coefficient is uniform for fully developed flow
- Duct length is unknown
- The fluid is air

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(2) Problem Definition

(i) Find the required length

(ii) Determine surface temperature at outlet

(3) Solution Plan

(i) Apply conservation of energy

(ii) Compute the Reynolds

(iii) Apply constant surface solution

(iv) Use Table 6.2 for h

(4) Plan Execution

(i) Assumptions

- Steady state

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- Constant properties
- Uniform surface flux
- Negligible changes in kinetic and potential energy
- Negligible axial conduction
- Negligible dissipation

(ii) Analysis

Conservation of energy

$$P L q_s'' = \dot{m} c_p (T_{mo} - T_{mi}) \quad \text{(a)}$$

c_p = specific heat, J/kg$-^\circ$C

L = channel length, m

$\dot{m}$ = mass flow rate, kg/s

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P = perimeter, m

q_s'' = surface heat flux = 590 W/m^2

T_{mi} = 40°C

T_{mo} = 120°C

Solve (a) for L

$$L = \frac{\dot{m} c_p (T_{mo} - T_{mi})}{P q''} \quad \text{(b)}$$

Find $\dot{m}$ and P

$$\dot{m} = \rho S^2 \bar{u} \quad \text{(c)}$$

$$P = 4S \quad \text{(d)}$$

S = duct side = 0.04 m

$\bar{u}$ = mean flow velocity = 0.32 m/s

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ρ = density, kg/m^3

(c) and (d) into (b)

$$L = \frac{\rho S \bar{u} c_p (T_{mo} - T_{mi})}{4 q''} \quad \text{(e)}$$

Surface temperature: Use solution (6.10)

$$T_s(x) = T_{mi} + q_s'' \left(\frac{Px}{\dot{m} c_p} + \frac{1}{h(x)} \right) \quad \text{(f)}$$

$h(x)$ = local heat transfer coefficient, W/m^2$-^\circ$C

$T_s(x)$ = local surface temperature, $^\circ$C

x = distance from inlet, m

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Maximum surface temperature at $x = L$

$$T_s(L) = T_{mi} + q_s''\left(\frac{4L}{\rho S \bar{u} c_p} + \frac{1}{h(L)}\right) \quad \text{(g)}$$

Determine $h(L)$: Compute the Reynolds number

$$Re_{De} = \frac{\bar{u} D_e}{\nu} \quad \text{(h)}$$

D_e = equivalent diameter, m

ν = kinematic viscosity, m^2/s

$$D_e = 4\frac{A}{P} = 4\frac{S^2}{4S} = S \quad \text{(i)}$$

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(i) into (h)

$$Re_{De} = \frac{\bar{u} S}{\nu} \quad \text{(j)}$$

Properties: At $\bar{T}_m$

$$\bar{T}_m = \frac{T_{mi} + T_{mo}}{2} \quad \text{(k)}$$

$$\bar{T}_m = \frac{(40+120)(^\circ\text{C})}{2} = 80^\circ\text{C}$$

Properties:

$c_p = 1009.5$ J/kg$-^\circ$C

$k = 0.02991$ W/m$-^\circ$C

$Pr = 0.706$

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$\nu = 20.92 \times 10^{-6}$ m^2/s

$\rho = 0.9996$ kg/m^3

(j) gives

$$Re_{De} = \frac{0.32(\text{m/s})0.04(\text{m})}{20.92\times10^{-6}(\text{m}^2/\text{s})} = 611.9,$$ **laminar flow**

(6.55) and Table 6.2

$$\overline{Nu}_{De} = \frac{\bar{h} D_e}{k} = 3.608 \quad \text{(l)}$$

$h = \bar{h}$

$$\bar{h} = 3.608\frac{k}{D_e} \quad \text{(m)}$$

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(iii) Computations. Use (e)

$$L = \frac{0.9996(\text{kg/m}^3)\,0.04(\text{m})0.32(\text{m/s})\,1009.5(\text{J/kg-}^\circ\text{C})(120-40)(^\circ\text{C})}{(4)\,590(\text{W/m}^2)} = 0.4378 \text{ m}$$

Use (m)

$$h(L) = \bar{h} = 3.608\frac{0.02991(\text{W/m}-^\circ\text{C})}{0.04(\text{m})} = 2.7\ \text{W/m}^2-^\circ\text{C}$$

Substitute into (g)

$$T_s(L) = 40(^\circ\text{C}) + 590(\text{W/m}^2)\left(\frac{4(0.4378)(\text{m})}{0.9996(\text{kg/m}^3)\,0.04(\text{m})0.32\,(\text{m/s})\,1009.5(\text{J/kg}-^\circ\text{C})} + \frac{1}{2.7(\text{W/m}^2-^\circ\text{C})}\right)$$

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$T_s(L) = 338.5\,^\circ\text{C}$

(iv) Checking. ***Dimensional check***:

Quantitative checks: (1) Alternate approach to determining : Newton's law at outlet

$$q''_s = h[T_s(L) - T_{mo}] \quad \text{(n)}$$

$T_s(L)$ Solve for

$$T_s(L) = T_{mo} = \frac{q''_s}{h} = 120(^\circ\text{C}) + \frac{590(\text{W/m}^2)}{2.7(\text{W/m}^2-^\circ\text{C})} = 338.5^\circ\text{C}$$

(2) Compare h with Table 1.1

Limiting check: $L=0$ for $T_{mo} = T_{mi}$. Set $T_{mo} = T_{mi}$ into (e) gives $L = 0$

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(5) Comments

(i) Maximum surface temperature is determined by the heat transfer coefficient at outlet

(ii) Compute the Reynolds number to establish if the flow is laminar or turbulent and if it is developing or fully developed

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6.8 Thermal Entrance Region: Laminar Flow Through Tubes

6.8.1 Uniform Surface Temperature: Graetz Solution

- Laminar flow through tube
- Velocity is fully developed
- Temperature is developing
- No axial conduction (Pe > 100)
- Uniform surface temperature T_s

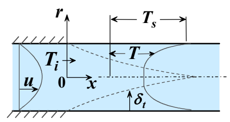

Fig.6.8

Velocity:

$$v_r = 0 \quad (3.1)$$

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$$v_z = \frac{1}{4\mu}\frac{dp}{dz}(r^2 - r_o^2) \quad (3.12)$$

Rewrite (3.1)

$$v_z^* = \frac{v_z}{\bar{u}} = 2\left[1 - R^2\right] \quad (6.61)$$

(3.1) and (6.61) into (6.31)

$$\frac{1}{2}\left(1 - R^2\right)\frac{\partial\theta}{\partial\xi} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\theta}{\partial R}\right) \quad (6.62)$$

Boundary conditions:

$$\frac{\partial\theta(\xi,0)}{\partial R} = 0$$

$$\theta(\xi,1) = 0$$

$$\theta(0,R) = 1$$

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Solution summary: Assume a product solutions

$$\theta(\xi,R) = X(\xi)R(R) \quad (a)$$

(a) into (6.62), separating variables

$$\frac{dX_n}{d\xi} + 2\lambda_n^2 X_n = 0 \quad (b)$$

$$\frac{d^2R_n}{dR^2} + \frac{1}{R}\frac{dR_n}{dR^2} + \lambda_n^2(1 - R^2)R_n^2 = 0 \quad (c)$$

- λ_n = eigenvalues obtained from the boundary conditions
- Solution $X_n(\xi)$ to (b) is exponential
- Solution $R_n(R)$ to (c) is not available in terms of simple tabulated functions

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Solutions to (b) and (c) into (a)

$$\theta(\xi, R) = \sum_{n=0}^{\infty} C_n R_n(R) \exp(-2\lambda_n^2 \xi) \qquad (6.64)$$

C_n = constant

Surface flux:

$$q_s''(\xi) = \frac{k}{r_o}(T_s - T_i)\frac{\partial \theta(\xi,1)}{\partial R} \qquad (6.22)$$

(6.64) gives

$$\frac{\partial \theta(\xi,1)}{\partial R} = \sum_{n=0}^{\infty} C_n \frac{dR_n(1)}{dR_n} \exp(-2\lambda_n^2 \xi) \qquad (d)$$

Define

$$G_n = -\frac{C_n}{2}\frac{dR_n(1)}{dR} \qquad (e)$$

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(e) into (6.22)

$$q_s''(\xi) = -\frac{2k}{r_o}(T_s - T_i)\sum_{n=0}^{\infty} G_n \exp(-2\lambda_n^2 \xi) \qquad (6.65)$$

Local Nusselt number: is given by

$$Nu(\xi) = \frac{-2}{\theta_m(\xi)}\frac{\partial \theta(\xi,1)}{\partial R} \qquad (6.27)$$

(d) gives $\partial \theta(\xi,1)/\partial R$

Mean temperature $\theta_m(\xi)$: (6.61) and (6.64) into (6.32b), integrate by parts and use(e)

$$\theta_m(\xi) = 8\sum_{n=0}^{\infty} \frac{G_n}{\lambda_n^2} \exp(-2\lambda_n^2 \xi) \qquad (6.66)$$

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(d), (e) and (6.66) into (6.27)

$$Nu(\xi) = \frac{\sum_{n=0}^{\infty} G_n \exp(-2\lambda_n^2 \xi)}{2\sum_{n=0}^{\infty} \frac{G_n}{\lambda_n^2} \exp(-2\lambda_n^2 \xi)} \qquad (6.87)$$

Average Nusselt number: For length x

$$\overline{Nu}(\xi) = \frac{\overline{h}(\xi) D}{k} \qquad (f)$$

Two methods for determining $\overline{h}(\xi)$:

(1) Integrate local $h(\xi)$ to obtain $\overline{h}(\xi)$

(2) Use (6.13)

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$$T_m(x) = T_s + (T_i - T_s)\exp[-\frac{P\bar{h}}{m\,c_p}x] \quad (6.13)$$

Solve for $\bar{h}$

$$\bar{h} = -\frac{m\,c_p}{Px}\ln\frac{T_m(x)-T_s}{T_i - T_s} \quad (g)$$

(g) into (f), use $m = \rho\bar{u}\,\pi\,D^2/4$ $P = \pi\,D$ and definitions of ξ, Re_D and θ_m in (6.21) and (6.25)

$$\overline{Nu}(\xi) = -\frac{1}{4\xi}\ln\theta_m(\xi) \quad (6.68)$$

- Need λ_n and G_n to compute $q_s''(\xi), \theta_m(\xi),\ Nu(\xi)$ and $\overline{Nu}(\xi)$
- Table 6.3 gives λ_n and G_n
- (6.67) and (6.68) are plotted in Fig. 6.9 as $Nu(\xi)$ and $\overline{Nu}(\xi)$

Table 6.3
Uniform surface temperature [4]

n	λ_n	G_n
0	2.70436	0.74877
1	6.67903	0.54383
2	10.67338	0.46286
3	14.67108	0.41542
4	18.66987	0.38292
5	22.66914	0.35869
6	26.66866	0.33962
7	30.66832	0.32406
8	34.66807	0.31101
9	38.66788	0.29984
10	42.66773	0.29012

Table 6.4
Local and average Nusselt number for tube at uniform surface temperature [5]

$\xi = \frac{x/D}{Re_D Pr}$	$Nu(\xi)$	$\overline{Nu}(\xi)$
0	∞	∞
0.0005	12.8	19.29
0.002	8.03	12.09
0.005	6.00	8.92
0.02	4.17	5.81
0.04	3.77	4.86
0.05	3.71	4.64
0.1	3.66	4.15
∞	3.66	3.66

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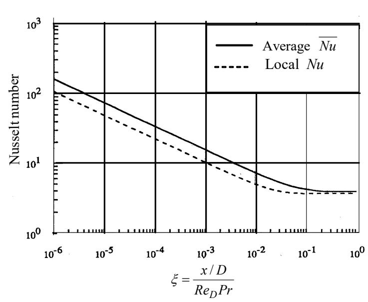

Fig. 6.9 Local and average Nusselt number for tube at uniform surface temepratu re [4]

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Comments

(1) $\overline{Nu}_D$ and Nu_D decrease with distance from entrance

(2) At any location ξ $\overline{Nu}_D > Nu_D$

(3) Asymptotic value (at $\xi \approx 0.05$) for $\overline{Nu}_D$ and Nu_D is 3.657. Same result of fully develop analysis

$$Nu(\infty) = 3.657 \qquad (6.69)$$

(4) Properties at

$$\overline{T}_m = \frac{T_{mi} + T_{mo}}{2} \qquad (6.70)$$

(5) Solution by trial and error if T_{mo} is to be determined

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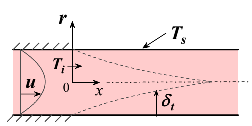
Example 6.5: Hot Water Heater

- Fully developed velocity in tube
- Developing temperature
- Uniform inlet temperature $T_i = 25^\circ$ C
- Diameter = 1.5 cm
- Length = 80 cm
- Flow rate = 0.002 kg/s
- Heat water to 75° C



Determine: Surface temperature

(1) Observations

- Uniform surface temperature
- Compute Reynolds number: Laminar or turbulent flow?

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- Compute L_h and L_t : Can they be neglected?

(2) Problem Definition

(i) Determine T_s

(ii) Determine h

(3) Solution Plan

(i) Apply uniform surface temperature results

(ii) Compute the Reynolds number: Establish if problem is entrance or fully developed

(iii) Use appropriate results for Nusselt number

(4) Plan Execution

(i) Assumptions

- Steady state

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- Constant properties
- Uniform surface temperature
- Negligible changes in kinetic and potential energy
- Negligible axial conduction
- Negligible dissipation

(ii) Analysis

Uniform surface temperature

$$T_m(x) = T_s + (T_{mi} - T_s)\exp[-\frac{P\bar{h}}{m\,c_p}x] \quad (6.13)$$

$\bar{h}$ = average heat transfer coefficient, $\text{W/m}^2-^{o}\text{C}$

m = flow rate $= 0.002\,\text{kg/s}$

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T_{mi} = mean inlet temperature $= 25^{o}\,\text{C}$

T_{mo} = mean outlet temperature $= 75^{o}\,\text{C}$

Apply (6.13) at outlet ($x = L$) and solve for T_s

$$T_s = \frac{1}{1 - \exp(P\bar{h}L/mc_p)}\left[T_{mi} - T_m(L)\exp(P\bar{h}L/mc_p)\right] \quad (a)$$

Properties: at $\bar{T}_m$

$$\bar{T}_m = \frac{T_{mi} + T_{mo}}{2}$$

Perimeter P

$$P = \pi D$$

D = diameter = 1.5 cm = 0.015 m

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Determine $\bar{h}$: Compute the Reynolds

$$Re_D = \frac{\bar{u}D}{\nu}$$

$$\bar{u} = \frac{4m}{\rho\pi D^2} \quad (e)$$

Properties: at $\bar{T}_m$

$$\bar{T}_m = \frac{(20+80)(^{o}\text{C})}{2} = 50^{o}\,\text{C}$$

c_p = 4182 J/kg-ºC

k = 0.6405 W/m-ºC

Pr = 3.57

ν = 0.5537×10-6 m²/s

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ρ = 988 kg/m3

Use (e)

$$\bar{u} = \frac{4(0.002)(\text{kg/s})}{988(\text{kg/m}^3)\pi\ (0.015)^2(\text{m}^2)} = 0.01146\ \text{m/s}$$

Use (d) gives

$$Re_D = \frac{0.01146(\text{m/s})0.015(\text{m})}{0.5537\times10^{-6}(\text{m}^2/\text{s})} = 310.5$$, laminar flow

Determine L_h and L_t

$$\frac{L_h}{D} = C_h Re_D \quad (6.5)$$

$$\frac{L_t}{D} = C_t Pr Re_D \quad (6.6)$$

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C_h= 0.056 (Table 6.1)

C_t = 0.033 (Table 6.1)

(6.5) and (6.6)

L_h = 0.056 × 0.015 (m) × 310.5 = 0.26 m

L_t = 0.033 × 0.015 (m) × 310.5 × 3.57 = 0.55 m

- L_h and L_t are not negligible, tube length L = 0.8 m
- Use Graetz solution Fig. 6.9 or Table 6.4

Compute ξ

$$\xi = \frac{x/D}{Re_D Pr} \quad (f)$$

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Nusselt number $\overline{Nu}$ gives $\bar{h}$

$$\bar{h} = \frac{k}{D}\overline{Nu} \quad (g)$$

(iii) Computation. Evaluating ξ at $x = L$

$$\xi = \frac{0.8(\text{m})/).015(\text{m})}{310.5\times3.57} = 0.0481$$

At $\xi = 0.481$ Fig. 6.9 gives

$$\overline{Nu} \approx 4.6$$

Substitute into (g)

$$\bar{h} = \frac{0.6405(\text{W/m}-^{\text{o}}\text{C})}{0.015(\text{m})}4.6 = 196.4\text{W/m}^2-^{\text{o}}\text{C}$$

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Equation (a) gives T_s

Compute the exponent of the exponential in (a)

$$\frac{P\bar{h}L}{\dot{m}c_p} = \frac{\pi(0.015)(\text{m})(196.4(\text{W/m}^2-^{\circ}\text{C})0.8(\text{m})}{0.002(\text{kg/s})4182(\text{J/kg}-^{\circ}\text{C})} = 0.88524$$

Substitute into (a)

$$T_s = \frac{1}{1-\exp(0.88524)}\left[25(^{\circ}\text{C}) - 75(^{\circ}\text{C})\exp(0.88524)\right] = 110.1^{\circ}\text{C}$$

(iv) Checking. *Dimensional check*:

Limiting checks:

(i) For $T_{mi} = T(L)$ (no heating) T_s should be equal to T_{mi} .

Set $T_{mi} = T(L)$ in (a) gives $T_s = T_{mi}$

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(ii) If $L = 0$, T_s should be infinite. Set in $L = 0$ (a) gives $T_s = \infty$

Quantitative checks:

(i) Approximate check:

$$\textit{Energy added at the surface} = \textit{Energy gained by water} \quad \text{(h)}$$

Let

$\bar{T}_m$ = average water temperature in tube

$$\text{Energy added at surface} = \bar{h}\pi DL(T_s - \bar{T}_m) \quad \text{(i)}$$

$$\text{Energy gained by water} = \dot{m}c_p(T_{mo} - T_{mi}) \quad \text{(j)}$$

(j) and (k) into (i), solve for T_s

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$$T_s = \bar{T}_m + \frac{\dot{m}\, c_p(T_{mo} - T_{mi})}{\bar{h}\,\pi\, DL} \quad \text{(k)}$$

(k) gives

$$T_s = 50(^{\circ}\text{C}) + \frac{0.002(\text{kg/s})4182(\text{J/kg}-^{\circ}\text{C})(75-25)(^{\circ}\text{C})}{196.4(\text{W/m}^2-^{\circ}\text{C})\pi(0.0155(\text{m})(0.8)(\text{m})} = 106.5^{\circ}\text{C}$$

(ii) Compare computed $\bar{h}$ with Table 1.1

(5) Comments

- Small error is due to reading Fig. 6.9
- Fully developed temperature model:

$$Nu_D = 3.657, \text{ gives } \bar{h} = 156.3\text{W/m}^2-^{\circ}\text{C}$$

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6.8.2 Uniform Surface Heat Flux

- Repeat Graetz entrance problem with uniform surface heat
- Fully developed inlet velocity
- Laminar flow through tube
- Temperature is developing
- No axial conduction (Pe > 100)

Energy equation: Same as for Graetz problem

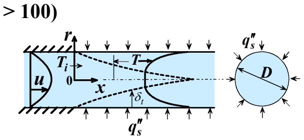

Fig. 6.10

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$$\frac{1}{2}\left(1-R^2\right)\frac{\partial\theta}{\partial\xi}=\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\theta}{\partial R}\right) \quad (6.62)$$

Boundary conditions:

$$\frac{\partial\theta(\xi,0)}{\partial R}) = 0 \quad (6.71a)$$

$$\frac{\partial\theta(\xi,1)}{\partial R}=\frac{q''_s r_o}{k(T_i-T_s)} \quad (6.71b)$$

$$\theta(0,R)=1 \quad (6.71c)$$

Analytic solutions: Based on separation of variables

(1) Local Nusselt number

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$$Nu(\xi)=\frac{hx}{k}=\left[\frac{11}{48}-\frac{1}{2}\sum_{n=1}^{\infty}A_n\exp(-2\beta_n^2\xi)\right]^{-1} \quad (6.72)$$

Table 6.6 lists eigenvalues β_n^2 and constant A_n

(2) Average Nusselt number

$$\overline{Nu}(\xi)=\frac{hx}{k}=\left[\frac{11}{48}-\frac{1}{2}\sum_{n=1}^{\infty}A_n\frac{1-\exp(-2\beta_n^2\xi)}{2\beta_n^2\xi}\right]^{-1} \quad (6.73)$$

Limiting case:

Fully developed: Set $\xi=\infty$ (6.72) or (6.73)

$$Nu(\infty)=\left(\frac{11}{48}\right)^{-1}=4.364 \quad (6.74)$$

Same as fully developed result (6.55).

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Graphical results: Fig. 6.11

Table 6.5 Uniform surface flux [4]		
n	β_n^2	A_n
1	25.6796	0.198722
2	83.8618	0.069257
3	174.1667	0.036521
4	296.5363	0.023014
5	450.9472	0.016030
6	637.3874	0.011906
7	855.8495	0.009249
8	1106.3290	0.007427
9	1388.8226	0.006117
10	1703.3279	0.005141

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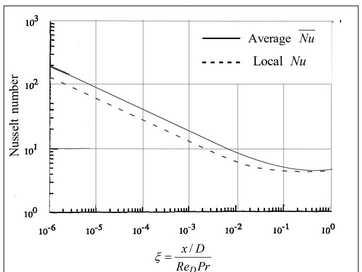

Fig. 6.11 Local and average Nusselt number for tube at uniform surface heat flux [4]

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