CHAPTER 7

FREE CONVECTION

7.1 Introduction

- **Applications:**
  Solar collectors
  Pipes
  Ducts
  Electronic packages
  Walls and windows

7.2 Features and Parameters of Free Convection

(1) Driving Force: Natural

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Requirements: (i) Acceleration field, (ii) Density gradient

Temperature gradient → density gradient

(2) Governing Parameters: Two:

(i) Grashof number Gr_L

$$Gr_L = \frac{\beta g(T_s - T_\infty)L^3}{\nu^2} \quad (7.1)$$

β = coefficient of thermal expansion (compressibility), a property

$$\beta = \frac{1}{T} \quad \text{for ideal gas} \quad (2.21)$$

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(ii) Prandtl number Pr

$$Pr = \frac{c_p \mu}{k}$$

Rayleigh number Ra_L

$$Ra_L = Gr_L Pr = \frac{\beta g(T_s - T_\infty)L^3}{\nu^2} Pr = \frac{\beta g(T_s - T_\infty)L^3}{\nu \alpha} \quad (7.2)$$

(3) Boundary Layer: laminar, turbulent or mixed

Criterion: $Ra_x > 10^4$

(4) Transition: Laminar to Turbulent Flow

Criterion for vertical plates: Transition Rayleigh number

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$$Ra_{x_t} \approx 10^9$$

(5) External vs. Enclosure free convection:

(i) External: over:

Vertical surfaces
Inclined surfaces
Horizontal cylinders
Spheres

(ii) Enclosure: in:

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Rectangular confines
Concentric cylinders
Concentric spheres

(6) Analytic Solution

Requires simultaneous integration of continuity, momentum and energy

7.3 Governing Equations

Approximations:

(1) Density is constant except in gravity forces

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(2) Boussinesq approximation: relate density change to temperature change

(3) Negligible dissipation

Assume:
Steady state
Two-dimensional
Laminar

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (7.1)$$

***x*-momentum:**

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$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\beta g(T-T_\infty)-\frac{1}{\rho_\infty}\frac{\partial}{\partial x}(p-p_\infty)+\nu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}) \qquad (7.2)$$

y-momentum:

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho_\infty}\frac{\partial}{\partial y}(p-p_\infty)+\nu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}) \qquad (7.3)$$

Energy:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \qquad (7.4)$$

NOTE:

(1) Gravity points in the negative *x*-direction

(2) Flow and temperature fields are coupled

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7.3.1 Boundary Layer Equations

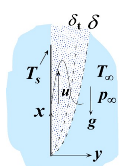

Fig. 7.1

- Velocity and temperature boundary layers
- Apply approximation used in forced convection

y-component of the Navier-Stokes equations reduces to

$$\frac{\partial}{\partial y}(p-p_\infty)=0 \qquad (a)$$

External flow: Neglect ambient pressure variation in *x*

$$\frac{\partial}{\partial x}(p-p_\infty)=0 \qquad (b)$$

Furthermore, for boundary layer flow

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$$\frac{\partial^2 u}{\partial x^2} << \frac{\partial^2 u}{\partial y^2} \qquad (c)$$

x-momentum:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=+\beta g\left(T-T_\infty\right)+\nu\frac{\partial^2 u}{\partial y^2} \qquad (7.5)$$

Neglect axial conduction

$$\frac{\partial^2 T}{\partial x^2} << \frac{\partial^2 T}{\partial y^2} \qquad (d)$$

Energy: (7.4) becomes

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \qquad (7.6)$$

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7.4 Laminar Free Convection over a Vertical Plate: Uniform Surface Temperature

- Vertical plate
- Uniform temperature T_s
- Infinite fluid at temperature T_∞

Determine: velocity and temperature distribution

7.4.1 Assumptions

(1) Steady state
(2) Laminar flow
(3) Two-dimensional

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(1) Constant properties
(2) Boussinesq approximation
(3) Uniform surface temperature
(4) Uniform ambient temperature
(5) Vertical plate
(9) Negligible dissipation

7.4.2 Governing Equations

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (7.1)$$

***x*-momentum:**

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$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = +\beta g\left(T - T_\infty\right) + \nu\frac{\partial^2 u}{\partial y^2} \qquad (7.5)$$

Energy:

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha\frac{\partial^2 \theta}{\partial y^2} \qquad (7.7)$$

where θ is a dimensionless temperature defined as

$$\theta = \frac{T - T_\infty}{T_s - T_\infty} \qquad (7.8)$$

7.4.3 Boundary Conditions

Velocity:

(1) $u(x,0) = 0$

(2) $v(x,0) = 0$

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(3) $u(x,\infty) = 0$

(4) $u(0,y) = 0$

Temperature:

(5) $\theta(x,0) = 1$

(6) $\theta(x,\infty) = 0$

(7) $\theta(0,y) = 0$

7.4.4 Similarity Transformation

- Transform three PDE to two ODE
- Introduce similarity variable $\eta(x,y)$

$$\eta(x,y) = C\frac{y}{x^{1/4}} \tag{7.8}$$

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$$C = \left[\frac{\beta g(T_s - T_\infty)}{4\nu^2}\right]^{1/4} \tag{7.9}$$

(7.9) into (7.8)7

$$\eta = \left(\frac{Gr_x}{4}\right)^{1/4}\frac{y}{x} \tag{7.10}$$

Local Grashof number:

$$Gr_x = \frac{\beta g(T_s - T_\infty)x^3}{\nu^2} \tag{7.11}$$

Let

$$\theta(x,y) = \theta(\eta) \tag{7.12}$$

Stream function ψ satisfies continuity

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$$u = \frac{\partial \psi}{\partial y} \tag{7.14}$$

Using Blasius solution as a guide, ψ for this problem is

$$\psi == 4\nu\left(\frac{Gr_x}{4}\right)^{\frac{1}{4}}\xi(\eta) \tag{7.15}$$

$\xi(\eta)$ is an unknown function

Introduce (7.15) into (7.13) and (7.14)

$$u = 2\nu\frac{\sqrt{Gr_x}}{x}\frac{d\xi}{d\eta} \tag{7.16}$$

$$v = \frac{\nu}{(4)^{1/4}}\frac{(Gr_x)^{1/4}}{x}\left[\eta\frac{d\xi}{d\eta} - 3\xi\right] \tag{7.17}$$

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Combining (7.5), (7.7), (7.8), (7.12), (7.16), (7.17)

$$\frac{d^3\xi}{d\eta^3} + 3\xi\frac{d^2\xi}{d\eta^2} - 2\left(\frac{d\xi}{d\eta}\right)^2 + \theta = 0 \qquad (7.18)$$

$$\frac{d^2\theta}{d\eta^2} + 3Pr\xi\frac{d\theta}{d\eta} = 0 \qquad (7.19)$$

NOTE

- x and y are eliminated (7.18) and (7.19)
- Single independent variable η

Transformation of boundary conditions:

Velocity:

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(1) $\frac{d\xi(0)}{d\eta} = 0$

(2) $\xi(0) = 0$

(3) $\frac{d\xi(\infty)}{d\eta} = 0$

(4) $\frac{d\xi(\infty)}{d\eta} = 0$

Temperature:

(1) $\theta(0) = 1$

(2) $\theta(\infty) = 0$

(3) $\theta(\infty) = 0$

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NOTE:

(1) Three PDE are transformed into two ODE

(3) Five BC are needed

(4) Seven BC are transformed into five

(5) One parameter: Prandtl number.

7.4.5 Solution

- (7.18) and (7.19) are solved numerically
- Solution is presented graphically
- Figs. 7.2 gives $u(x,y)$
- Fig. 7.3 gives $T(x,y)$

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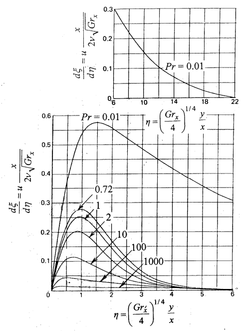

Fig. 7.2 Axial velocity distribution [1]

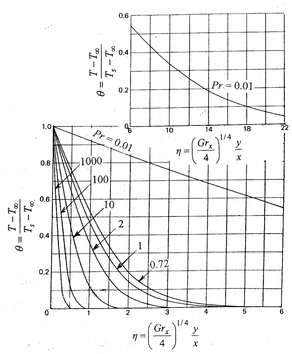

Fig. 7.3 Temperature distribution [1]

7.4.5 Heat Transfer Coefficient and Nusselt Number

Start with

$$h = \frac{-k \dfrac{\partial T(x,0)}{\partial y}}{T_s - T_\infty} \qquad (7.20)$$

Express in terms of θ and η

$$h = \frac{-k}{T_s - T_\infty} \frac{dT}{d\theta} \frac{d\theta(0)}{d\eta} \frac{\partial \eta}{\partial y}$$

Use (7.8) and (7.10)

$$h = \frac{-k}{x} \left[\frac{Gr_x}{4} \right]^{1/4} \frac{d\theta(0)}{d\eta} \qquad (7.21)$$

Define: local Nusselt number:

$$Nu_x = \frac{hx}{k} = -\left[\frac{Gr_x}{4}\right]^{1/4} \frac{d\theta(0)}{d\eta} \qquad (7.22)$$

Average heat transfer coefficient

$$\bar{h} = \frac{1}{L}\int_0^L h(x)dx \qquad (2.50)$$

(7.21) into (2.50), integrate

$$\bar{h} = -\frac{4}{3}\frac{k}{L}\left(\frac{Gr_L}{4}\right)^{1/4} \frac{d\theta(0)}{d\eta} \qquad (7.23)$$

Average Nusselt number

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$$\overline{Nu}_L = \frac{\bar{h}L}{k} = -\frac{4}{3}\frac{k}{L}\left(\frac{Gr_L}{4}\right)^{1/4} \frac{d\theta(0)}{d\eta} \qquad (7.24)$$

$\dfrac{d\theta(0)}{d\eta}$

- Is key factor in solution
- Depends on Prandtl number
- Values are listed in Table 7.1
- Obtained from numerical solution

Table 7.1 [1,2]

Pr	$-\dfrac{d\theta(0)}{d\eta}$	$\dfrac{d^2\xi(0)}{d\eta^2}$
0.01	0.0806	0.9862
0.03	0.136	
0.09	0.219	
0.5	0.442	
0.72	0.5045	0.676
0.733	0.508	0.6741
1.0	0.5671	0.6421
1.5	0.6515	
2.0	0.7165	0.5713
3.5	0.8558	
5.0	0.954	
7.0	1.0542	
10	1.1649	0.4192
100	2.191	0.2517
1000	3.9660	0.1450

- $\dfrac{d^2\xi(0)}{d\eta^2}$ in Table 7.1 gives surface velocity gradient and shearing stress

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Special Cases

$$Nu_x = 0.600(PrRa_x)^{1/4}, \quad Pr \rightarrow 0 \qquad (7.24)$$

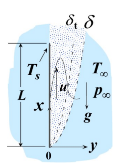
$$Nu_x = 0.503(PrGr_x)^{1/4}, \quad Pr \rightarrow \infty \qquad (7.25)$$

Example 7.1: Vertical Plate at Uniform Surface Temperature

- $8cm \times 8cm$ *vertical plate in air at 10 °C*
- *Uniform surface temperature = 70 °C*
- *Determine the following at* $x = L = 8$ *cm*:

  (1) *u at y = 0.2 cm*, (2) *T at y = 0.2 cm*

  (3) δ, (4) δ_t



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(5) Nu_L, (6) $h(L)$

(7) $q_s''(L)$, (8) q_T

Solution

(1) Observations

- External free convection
- Vertical plate
- Uniform surface temperature
- Check Rayleigh number for laminar flow
- If laminar, Fig. 7.2 for $u(x,y)$ and Fig. 7.3 $T(x,y)$ and δ_t
- Determine local Nu and h at $x = L$

(2) Problem Definition

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Determine flow and heat transfer characteristics for free convection over a vertical flat plate at uniform surface temperature.

(3) Solution Plan

- Laminar flow? Compute Rayleigh number
- If laminar, use Figs. 7.2 and 7.3.
- Use solution for Nu and h

(4) Plan Execution

(i) Assumptions

(1) Newtonian fluid
(2) Steady state
(3) Boussinesq approximations
(4) Two-dimensional

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(5) Laminar flow ($Ra_x < 10^9$)
(6) Flat plate
(7) Uniform surface temperature
(8) No dissipation
(9) No radiation.

(ii) Analysis and Computation

Compute the Rayleigh number:

$$Ra_L = \frac{\beta g (T_s - T_\infty) L^3}{\nu \alpha} \tag{7.2}$$

g = gravitational acceleration = 9.81 m/s^2

L = plate length = 0.08 m

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Ra_L = Rayleigh number at the trailing end $x = L$

T_s = surface temperature = 70^{o}C

T_∞ = ambient temperature = 10^{o}C

α = thermal diffusivity, m^2/s

β = coefficient of thermal expansion $= 1/T_f$ K^{-1}

ν = kinematic viscosity, m^2/s

Properties at T_f

$$T_f = \frac{T_s + T_\infty}{2} = \frac{(70+10)^{o}\text{C}}{2} = 40^{o}\text{C}$$

k = thermal conductivity = 0.0271 W/m$-^{o}$C

$Pr = 0.71$

$\nu = 16.96 \times 10^{-6}\, m^2/s$

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$$\alpha = \frac{\nu}{Pr} = \frac{16.96\times10^{-6}\,\text{m}^2/\text{s}}{0.71} = 23.89\times10^{-6}\,\text{m}^2/\text{s}$$

$$\beta = \frac{1}{40^{o}\text{C} + 273.13} = 0.0031934\text{K}^{-1}$$

Substituting into (7.2)

$$Ra_L = \frac{0.0031934(\text{K}^{-1})9.81(\text{m/s}^2)(70-10)(\text{K})(0.08)^3(\text{m}^3)}{16.96\times10^{-6}(\text{m}^2/\text{s})23.89\times10^{-6}(\text{m}^2/\text{s})} = 2.3752\times10^{6}$$

Thus the flow is laminar

(1) Axial velocity *u*:

$$u = 2\nu\frac{\sqrt{Gr_x}}{x}\frac{d\xi}{d\eta} \qquad (7.10)$$

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Fig. 7.2: $\frac{d\xi}{d\eta}$ vs. η

$$\eta = \left(\frac{Gr_x}{4}\right)^{1/4}\frac{y}{x} \qquad (a)$$

$$Gr_x = Gr_L = \frac{Ra_L}{Pr} = \frac{2.3792\times10^{6}}{0.71} = 3.351:$$

Use (a), evaluate η at $x = 0.08$ m and $y = 0.002$ m

$$\eta = \left(\frac{3.351\times10^{6}}{4}\right)^{1/4}\frac{0.0032\,(\text{m})}{0.08(\text{m})} = 1.21$$

Fig. 7.2, at $\eta = 1.21$ and $Pr = 0.71$, gives

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$$\frac{d\xi}{d\eta} = u\frac{x}{2\nu\sqrt{Gr_x}} \approx 0.27$$

Solve for u

$$u = 0.27\frac{2\nu\sqrt{Gr_L}}{L} = 0.27\frac{2(16.96\times10^{-6})(\text{m}^2/\text{s})\sqrt{3.351\times10^6}}{0.08(\text{m})} = 0.2096\,\text{m/s}$$

(1) Temperature T: At $\eta = 1.21$ and $Pr = 0.71$, Fig. 7.3 gives

$$\theta = \frac{T - T_\infty}{T_s - T_\infty} \approx 0.43$$

$$T \approx T_\infty + 0.43(T_s - T_\infty) = 10(^\circ\text{C}) + 0.43(70-10)(^\circ\text{C}) = 35.8^\circ\text{C}$$

(3) Velocity B.L. thickness δ :

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At $y = \delta$, axial velocity $u \approx 0$, Fig. 7.2 gives

$$\eta(x,\delta) \approx 5 = \left(\frac{Gr_L}{4}\right)^{1/4}\frac{\delta}{L}$$

Solve for δ

$$\delta = 5(0.08)(\text{m})\left(\frac{4}{3.37321\times10^6}\right)^{1/4} = 0.0132\,\text{m} = 1.32\,\text{cm}$$

(4) Temperature B.L. thickness δ_t :

At $y = \delta_t$, $T \approx T_\infty$, $\theta \approx 0$. Fig. 7.3 gives

$$\eta(x,\delta_t) \approx 4.5 = \left(\frac{Gr_L}{4}\right)^{1/4}\frac{\delta}{L}$$

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Solve for δ_t

$$\delta_t = 4.5(0.08)(\text{m})\left(\frac{4}{3.3511\times10^6}\right)^{1/4} = 0.0119\,\text{m} = 1.19\,\text{cm}$$

(5) Local Nusselt number:

$$Nu_x = \frac{hx}{k} = -\left[\frac{Gr_x}{4}\right]^{1/4}\frac{d\theta(0)}{d\eta} \qquad (7.22)$$

Table 7.1 gives $\frac{d\theta(0)}{d\eta}$ at $Pr = 0.71$

$$\frac{d\theta(0)}{d\eta} = -0.5018$$

Nusselt Number: Use (7.22), evaluate at $x = L = 0.08$ m

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$$Nu_L = \frac{hL}{k} = -\left[\frac{Gr_L}{4}\right]^{1/4}\frac{d\theta(0)}{d\eta} = 0.5018\left[\frac{3.351\times10^6}{4}\right]^{1/4} = 15.18$$

(6) Local heat transfer coefficient: at $x = L = 0.08$ m

$$h(L) = \frac{k}{L}Nu_L = \frac{0.0271(\text{W/m}-^{\circ}\text{C})}{0.08(\text{m})}15.18 = 5.14\ \text{W/m}^2-^{\circ}\text{C}$$

(7) Heat flux: Newton's law gives

$$q_s'' = h(T_s - T_\infty) = 5.14(\text{W/m}-^{\circ}\text{C})(70-10)(^{\circ}\text{C}) = 308.4\ \text{W/m}^2$$

(8) Total heat transfer:

$$q_T = \bar{h}A(T_s - T_\infty)$$

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$$\bar{h} = -\frac{4}{3}\frac{k}{L}\left(\frac{Gr_L}{4}\right)^{1/4}\frac{d\theta(0)}{d\eta}$$

$$\bar{h} = -\frac{4}{3}\frac{(0.0271)(\text{W/m}-^{\circ}\text{C})}{0.08(\text{m})}\left[\frac{3.351\times10^6}{4}\right]^{1/4}(-0.5018) = 6.86\ \text{W/m}^2-^{\circ}\text{C}$$

$$q_T = 6.86(\text{W/m}^2-^{\circ}\text{C})0.08(\text{m})0.08(\text{m})(70-10)(^{\circ}\text{C}) = 2.63\ \text{W}$$

(iii) Checking

Dimensional check:

Units for u, T, δ, δ_t, Nu, h, q'' and q_T are consistent

Quantitative check:

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(i) h is within the range given in Table 1.1

(ii) $\delta > \delta_t$. This must be the case

(5) Comments

(i) Magnitudes of u and h are relatively small

(ii) $h(x) < \bar{h}(x)$

7.5 Laminar Free Convection over a Vertical Plate: Uniform Surface Heat Flux

- Vertical Plate
- Uniform surface heat flux
- Infinite fluid at temperature T_∞
- Assumptions: Same as Section 7.4

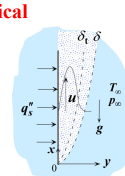

Fig. 7.4

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- Governing equations: Same as Section 7.4
- Boundary conditions: Replace uniform surface temperature with uniform surface flux

$$-k\frac{\partial T(x,0)}{\partial y}=q_s''$$

- Surface temperature is variable and unknown: $T_s(x)$
- Objective: Determine $T_s(x)$ and Nu_x
- Solution: Similarity transformation (Appendix F)
- Results:

(1) Surface temperature

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$$T_s(x)-T_\infty=-\left[5\frac{\nu^2(q_s'')^4}{\beta g k^4}x\right]^{1/5}\theta(0) \quad (7.27)$$

$\theta(0)$= constant, depends on Pr, given in Table 7.2

(2) Nusselt number

$$Nu_x=-\left[\frac{\beta g q_s''}{5\nu^2 k}x^4\right]^{1/5}\frac{1}{\theta(0)} \quad (7.28)$$

Table 7.2 [4]

Pr	$\theta(0)$
0.1	- 2.7507
1.0	- 1.3574
10	- 0.76746
100	- 0.46566

Correlation equation for $\theta(0)$ [5]

$$\theta(0)=-\left[\frac{4+9Pr^{1/2}+10Pr}{5Pr^2}\right]^{1/5} \quad (7.29)$$

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Example 7.2: Vertical Plate at Uniform Surface Flux

- $8cm\times 8cm$ *vertical plate in air at 10 °C*
- *Uniform surface flux = 308.4 °C*
- *Determine the following at* $x=2$, 4, 6 and *8 cm*

(1) Surface temperature

(2) Nusselt number

(3) Heat transfer coefficient

Solution

(1) Observations

- External free convection
- Vertical plate

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- Uniform surface heat flux
- Check Rayleigh number for laminar flow
- If laminar:

  Equation (7.27) gives $T_s(x)$

  Equation (7.28) gives Nu_x

  Nu_x gives $h(x)$

(2) Problem Definition

Determine $T_s(x)$, Nu_x and $h(x)$ for uniformly heated vertical plate in free convection

(3) Solution Plan

- Laminar flow? Compute Rayleigh number

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- Use (7.27) for $T_s(x)$
- Use (7.28) for Nu_x
- Nu_x gives $h(x)$

(4) Plan Execution

(i) Assumptions

(10) Newtonian fluid

(11) Steady state

(12) Boussinesq approximations

(13) Two-dimensional

(14) Laminar flow ($Ra_x < 10^9$)

(15) Flat plate

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(16) Uniform surface heat flux

(17) No dissipation

(18) No radiation

(ii) Analysis

Rayleigh number:

$$Ra_L = \frac{\beta g (T_s - T_\infty) L^3}{\nu \alpha} \qquad (7.2)$$

Surface temperature:

$$T_s(x) - T_\infty = -\left[5 \frac{\nu^2 (q_s'')^4}{\beta g k^4} x\right]^{1/5} \theta(0) \qquad (7.27)$$

Nusselt number:

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$$Nu_x = -\left[\frac{\beta g q_s''}{5\nu^2 k}x^4\right]^{1/5}\frac{1}{\theta(0)} \qquad (7.28)$$

Heat transfer coefficient

$$h(x) = \frac{k}{x}Nu_x \qquad (a)$$

Determine $\theta(0)$: Table 7.2 or equation (7.29)

$$\theta(0) = -\left[\frac{4 + 9Pr^{1/2} + 10Pr}{5Pr^2}\right]^{1/5} \qquad (7.29)$$

(iii) Computations

Properties at T_f

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$$T_f = \frac{\overline{T}_s + T_\infty}{2} \qquad (b)$$

where

$$\overline{T}_s = \frac{T_s(L) + T_s(0)}{2} \qquad (c)$$

$T_s(L)$ is unknown. Use iterative procedure:

(1) Assume $T_s(L)$
(2) Compute T_f
(3) Find properties at T_f
(4) Use (7.27) to calculate $T_s(L)$
(5) Compare with assumed value in step (1)
(6) Repeat until (1) and (5) agree

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Assume $T_s(L) = 130^\circ\mathrm{C}$

$$\overline{T}_s = \frac{T_s(L) + T_s(0)}{2} = \frac{(130+10)(^\circ\mathrm{C})}{2} = 70^\circ\mathrm{C}$$

$$T_f = \frac{T_s + T_\infty}{2} = \frac{(70+10)^\circ\mathrm{C}}{2} = 40^\circ\mathrm{C}$$

Properties of air at $40^\circ\mathrm{C}$:

$k = 0.0271\ \mathrm{W/m{-}^\circ C}$

$Pr = 0.71$

$\nu = 16.96 \times 10^{-6}\ \mathrm{m^2/s}$

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$$\alpha = \frac{\nu}{Pr} = \frac{16.96\times10^{-6}\,\mathrm{m^2/s}}{0.71} = 23.89\times10^{-6}\,\mathrm{m^2/s}$$

$$\beta = \frac{1}{40^{o}\mathrm{C} + 273.13} = 0.0031934\ \mathrm{K^{-1}}$$

Substitute into (7.2)

$$Ra_L = \frac{0.0031934\,(\mathrm{K^{-1}})9.81(\mathrm{m/s^2})(130-10)(\mathrm{K})(0.08)^3(\mathrm{m^3})}{16.96\times10^{-6}(\mathrm{m^2/s})23.89\times10^{-6}(\mathrm{m^2/s})}$$

$$= 4.7504\times10^{6}$$

- **Flow is laminar**

Use (7.9) to determine $\theta(0)$

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$$\theta(0) = -\left[\frac{4+9(0.71)^{1/2}+10(0.71)}{5(0.71)^2}\right]^{1/5} = -1.4928$$

Use (7.27) to compute $T_s(L)$

$$T_s(\mathrm{L}) = 10(^{o}\mathrm{C}) - \left[5\frac{(16.96\times10^{-6})^2(\mathrm{m^4/s^2})(308.4)^4(\mathrm{W^4/m^8})}{0.0031934(1/\mathrm{K})9.81(\mathrm{m/s^2})(0.0271)^4(\mathrm{W^4/m^4{-}K^4})}(0.1)(\mathrm{m})\right]^{1/5}(-1.4928) = 63.6^{o}\mathrm{C}$$

- **Computed** $T_s(L)$ **is lower than assumed value**
- **Repeat procedure until assume** $T_s(L) \approx$ **computed** $T_s(L)$

Result: $T_s(L) = 63.2^{o}\mathrm{C}$

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Thus,

$$T_f = 23.3^{o}\mathrm{C}$$

Properties of air $23.3^{o}\mathrm{C}$

$$k = 0.02601\,\mathrm{W/m{-}^{o}C}$$
$$Pr = 0.712525$$
$$\nu = 15.55\times10^{-6}\,\mathrm{m^2/s}$$
$$\alpha = 21.825\times10^{-6}\,\mathrm{m^2/s}$$
$$\beta = 0.003373\mathrm{K^{-1}}$$

(7.29) gives $\theta(0)$

$$\theta(0) = -1.4913$$

(7.27) gives $T_s(x)$

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$$T_s(L) = 10(^\circ\text{C}) -$$

$$\left[5\,\frac{(15.55\times10^{-6})^2(\text{m}^4/\text{s}^2)(308.4)^4(\text{W}^4/\text{m}^8)}{0.003373\,(1/\text{K})9.81\,(\text{m/s}^2)(0.02601)^4(\text{W}^4/\text{m}^4-\text{K}^4)}(\text{x}1)(\text{m})\right]^{1/5}\quad(-1.4913)$$

(7.28) gives Nu_x

$$Nu_x = -\left[\frac{0.003373(1/\text{K})9.81(\text{m/s}^2)308.4(\text{W/m}^2)}{5(15.55\times10^{-6})(\text{m}^2/\text{s})0.02601(\text{W/m}-^\circ\text{C})}(\text{x})^4(\text{m}^4)\right]^{1/5}\frac{1}{-1.4913} \quad \text{(e)}$$

(a) gives $h(x)$

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$$h(L) = \frac{k}{L}Nu_L = \frac{0.02601(\text{W/m}-^\circ\text{C})}{x(\text{m})}Nu_L \quad \text{(f)}$$

Use (d)-(f) to tabulate results at x = 0.02, 0.04, 0.06 and 0.08 m

x(m)	$T_s(x)(^\circ\text{C})$	Nu_x	$h(x)(\text{W/m}^2-^\circ\text{C})$	$q''_s(\text{W/m}^2)$
0.02	50.3	5.88	7.65	308.3
0.04	56.3	10.24	6.66	308.4
0.06	60.2	14.16	6.14	308.2
0.08	63.2	17.83	5.795	308.3

(iii) Checking

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Dimensional check:

Units for T_s, Nu and h are consistent.

Quantitative check:

(i) h is within the range given in Table 1.1

(ii) Compute q''_s using Newton's law gives uniform surface flux

(5) Comments

(i) Magnitude of h is relatively small

(ii) Surface temperature increases with distance along plate

7.6 Inclined Plates

Two cases:

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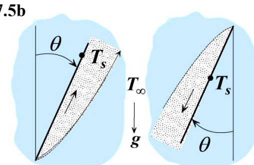
- Hot side is facing downward, Fig. 7.5a
- Cold side is facing upward, Fig. 7.5b
- Flow field: Same for both
- Gravity component = $g\cos\theta$
- **Solution:** Replace g by $g\cos\theta$ in solutions of Sections 7.4 and 7.5
- **Limitations**



$(a)\ T_s > T_\infty$ $(b)\ T_s < T_\infty$

Fig. 7.5

$$\theta \le 60^\circ$$

7.7 Integral Method

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- Obtain approximate solutions to free convection problems
- Example: Vertical plate at uniform surface temperature

7.7.1 Integral Formulation of Conservation of Momentum

- Simplifying assumption

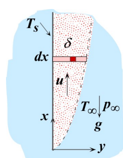

Fig 7.6

$$\delta = \delta_t \quad \text{valid for } Pr \approx 1 \tag{a}$$

Apply momentum theorem in x-direction to the element $\delta \times dx$

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$$\sum F_x = M_x(\text{out}) - M_x(\text{in}) \tag{b}$$

$\sum F_x$:

- Wall stress
- Pressure
- Gravity

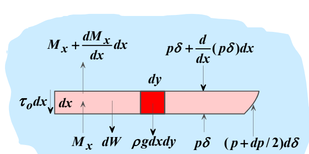

Fig. 7.7

$$p\delta + \left(p + \frac{dp}{2}\right)d\delta - p\delta - \frac{d}{dx}(p\delta)dx - \tau_o dx - = \left(M_x + \frac{dM_x}{dx}dx\right) - M_x \tag{c}$$

Simplify

$$-\delta dp - \tau_o dx - dW = \frac{dM_x}{dx}dx \tag{d}$$

Wall stress:

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$$\tau_o = \mu \frac{\partial u(x,0)}{\partial y} \qquad \text{(e)}$$

Weight:

- **Variable density**
- **Integrate weight of element $dx \times dy$**

$$dW = dx \int_0^{\delta} \rho g dy \qquad \text{(f)}$$

***x*-momentum:**

- **Constant density**

$$M_x = \rho \int_0^{\delta(x)} u^2 dy \qquad \text{(g)}$$

Substitute(e), (f) and (g) into (d)

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$$-\mu \frac{\partial u(x,0)}{\partial y} - \delta \frac{dp}{dx} - \int_0^{\delta} \rho g dy = \rho \frac{d}{dx} \int_0^{\delta} u^2 dy \qquad \text{(h)}$$

Combine pressure and gravity terms (B.L. flow)

$$\frac{dp}{dx} \cong \frac{dp_\infty}{dx} = -\rho_\infty g \qquad \text{(i)}$$

Rewrite

$$\delta \frac{dp}{dx} = -\rho_\infty g \delta = -\int_0^{\delta} \rho_\infty g dy \qquad \text{(j)}$$

(j) into (h)

$$-\mu \frac{\partial u(x,0)}{\partial y} + g \int_0^{\delta} (\rho_\infty - \rho) dy = \rho \frac{d}{dx} \int_0^{\delta} u^2 dy \qquad \text{(k)}$$

Density change:

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$$\rho_\infty - \rho = \rho\beta(T - T_\infty) \qquad \text{(2.28)}$$

(2.28) into (k), assume constant $\rho\beta$

$$-\nu \frac{\partial u(x,0)}{\partial y} + \beta g \int_0^{\delta} (T - T_\infty) dy = \frac{d}{dx} \int_0^{\delta} u^2 dy \qquad \text{(7.30)}$$

NOTE:

(1) no shearing force on the slanted surface

(2) (7.30) applies to laminar and turbulent flow

(3) (7.30) is a first order O.D.E.

7.7.2 Integral Formulation of Conservation of Energy

Assume:

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- No changes in kinetic and potential energy
- Neglect axial conduction
- Neglect dissipation
- Properties are constant

$$-\alpha \frac{\partial T(x,0)}{\partial y} = \frac{d}{dx}\int_0^{\delta(x)} u(T - T_\infty)dy \qquad (7.31)$$

NOTE:

Integral formulation of energy is the same for free convection and for forced convection

7.7.3 Integral Solution

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Since we assumed $\delta = \delta_t$,
either momentum or energy gives δ_t .
To satisfy both,
introduce another unknown in assumed velocity

- Vertical plate
- Uniform temperature T_s
- Quiescent fluid at uniform temperature T_∞

Solution Procedure for Forced Convection:

(1) Velocity is assumed in terms of $\delta(x)$

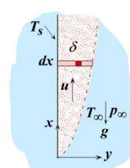

Fig 7.6

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(1) Momentum gives $\delta(x)$

(2) Temperature is assumed in terms $\delta_t(x)$

(3) Energy gives $\delta_t(x)$

Solution Procedure for Free Convection:

Since we assumed $\delta = \delta_t$,
either momentum or energy gives δ_t .
To satisfy both,
introduce another unknown in assumed velocity

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Assumed Velocity Profile

$$u(x,y) = a_0(x) + a_1(x)y + a_2(x)y^2 + a_3(x)y^3 \qquad \text{(a)}$$

Boundary conditions:

(1) $u(x,0) = 0$

(2) $u(x,\delta) \cong 0$

(3) $\dfrac{\partial u(x,\delta)}{\partial y} \cong 0$

(4) $\dfrac{\partial^2 u(x,0)}{\partial y^2} = -\dfrac{\beta g}{\nu}(T_s - T_\infty)$

- Setting $y = 0$ in the x-component, (7.2), to obtain condition (4)
- 4 boundary conditions give a_n

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(a) gives

$$u = \frac{\beta g(T_s - T_\infty)}{4\nu^2}\delta y\left[1 - 2\frac{y}{\delta} + \frac{y^2}{\delta^2}\right]$$

Rewrite as

$$u = \left[\frac{\beta g(T_s - T_\infty)}{4\nu^2}\delta^2\right]\frac{y}{\delta}\left[1 - \frac{y}{\delta}\right]^2 \qquad \text{(b)}$$

Introduce a second unknown in (b). Define

$$u_o(x) = \left[\frac{\beta g(T_s - T_\infty)}{4\nu^2}\delta^2\right] \qquad \text{(c)}$$

(b) becomes

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$$u = u_o(x)\,\frac{y}{\delta}\left[1 - \frac{y}{\delta}\right]^2 \qquad \text{(7.32)}$$

Assumed Temperature Profile

$$T(x,y) = b_0(x) + b_1(x)\,y + b_2(x)\,y^2 \qquad \text{(d)}$$

Boundary Conditions:

(1) $T(x,0) = T_s$

(2) $T(x,\delta) \cong T_\infty$

(3) $\dfrac{\partial T(x,\delta)}{\partial y} \cong 0$

- 3 boundary conditions give b_n

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(d) becomes

$$T(x,y) = T_\infty + (T_s - T_\infty)\left[1 - \frac{y}{\delta}\right]^2 \qquad (7.33)$$

Heat Transfer Coefficient and Nusselt Number

$$h = \frac{-k\dfrac{\partial T(x,0)}{\partial y}}{T_s - T_\infty} \qquad (7.20)$$

(7.33) into (7.20)

$$h = \frac{2k}{\delta(x)} \qquad (7.34)$$

Local Nusselt number:

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$$Nu_x = \frac{hx}{k} = 2\frac{x}{\delta(x)} \qquad (7.35)$$

Determine $\delta(x)$

Solution

Use momentum: substitute (7.32) and (7.33) into (7.30)

$$-\nu\frac{u_o}{\delta} + \beta g(T_s - T_\infty)\int_0^\delta \left[1 - \frac{y}{\delta}\right]^2 dy = \frac{d}{dx}\left\{\frac{u_o^2}{\delta^2}\int_0^\delta y^2\left[1 - \frac{y}{\delta}\right]^4 dy\right\} \qquad (e)$$

Evaluating the integrals

$$\frac{1}{105}\frac{d}{dx}\left[u_o^2\delta\right] = \frac{1}{3}\beta g(T_s - T_\infty)\delta - \nu\frac{u_o}{\delta} \qquad (7.36)$$

Use energy: substitute(7.32) and (7.33) into (7.31)

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$$2\alpha(T_s - T_\infty)\frac{1}{\delta} = (T_s - T_\infty)\frac{d}{dx}\left\{\frac{u_o}{\delta}\int_0^{\delta(x)} y\left[1 - \frac{y}{\delta}\right]^4 dy\right\} \qquad (f)$$

Evaluating the integrals

$$\frac{1}{60}\frac{d}{dx}\left[u_o\delta\right] = \alpha\frac{1}{\delta} \qquad (7.37)$$

- The two dependent variables are $\delta(x)$ and $u_o(x)$
- (7.36) and (7.37) are two simultaneous first order O.D.E.
- Assume a solution of the form

$$u_o(x) = Ax^m \qquad (7.38)$$

$$\delta(x) = Bx^n \qquad (7.39)$$

- Determine the constants A, B, m and n

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- **Substitute (7.38) and (7.39) into (7.36) and (7.37)**

$$\frac{2m+n}{105}A^2Bx^{2m+n-1} = \frac{1}{3}\beta g(T_o - T_\infty)Bx^n - \frac{A}{B}\nu x^{m-n} \qquad (7.40)$$

$$\frac{m+n}{210}ABx^{m+n-1} = \alpha\frac{1}{B}x^{-n} \qquad (7.41)$$

To satisfy (7.40) and (7.41) at all values of x, the exponents of x in each term must be identical

(7.40) requires

$$2m + n - 1 = n = m - n \qquad (g)$$

Similarly, (7.41) requires that

$$m + n - 1 = -n \qquad (h)$$

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Solve (g) and (h) for m and n gives

$$m = \frac{1}{2}, \quad n = \frac{1}{4} \qquad (i)$$

Introduce (i) into (7.40) and (7.41)

$$\frac{1}{85}A^2B = \frac{1}{3}\beta g(T_o - T_\infty)B - \frac{A}{B}\nu \qquad (j)$$

$$\frac{1}{280}AB = \alpha\frac{1}{B} \qquad (k)$$

Solve (j) and (k) for A and B

$$A = 5.17\nu\left[Pr + \frac{20}{21}\right]^{-1/2}\left[\frac{\beta g(T_s - T_\infty)}{\nu^2}\right]^{1/2} \qquad (l)$$

and

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$$B = 3.93\,Pr^{-1/2}\left(Pr + \frac{20}{21}\right)^{1/4}\left[\frac{\beta g(T_s - T_\infty)}{\nu^2}\right]^{-1/4} \qquad (m)$$

Substitute (i) and (m) into (7.39)

$$\frac{\delta}{x} = 3.93\left[\frac{20}{21}\frac{1}{Pr} + 1\right]^{1/4}(Ra_x)^{-1/4} \qquad (7.42)$$

Introduce(7.42) into (7.35)

$$Nu_x = 0.508\left[\frac{20}{21}\frac{1}{Pr} + 1\right]^{-1/4}(Ra_x)^{1/4} \qquad (7.43)$$

7.7.4 Comparison with Exact Solution for Nusselt Number

Exact solution:

$$Nu_x = -\left[\frac{Gr_x}{4}\right]^{1/4}\frac{d\theta(0)}{d\eta} \qquad (7.22)$$

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Rewrite above as

$$\left[\frac{Gr_x}{4}\right]^{-1/4} Nu_x = 0.508\left[\frac{20}{21}\frac{1}{Pr}+1\right]^{-1/4}(4Pr)^{1/4} \qquad (7.45)$$

Rewrite integral solution (7.43) as

$$\left[\frac{Gr_x}{4}\right]^{-1/4} Nu_x = -\frac{d\theta(0)}{d\eta} \qquad (7.44)$$

- **Compare right hand side of (7.45) with $-d\theta(0)/d\eta$ of exact solution (7.44)**
- **Comparison depends on the Prandtl number**

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- **Table 7.2 gives results**

Limiting Cases

(1) $Pr \to 0$

$$Nu_x\big|_{exact} = 0.600(PrRa_x)^{1/4}, \quad Pr \to 0 \qquad (7.25a)$$

$$Nu_x\big|_{integral} = 0.514(PrRa_x)^{1/4}, Pr \to 0 \qquad (7.46a)$$

(2) $Pr \to \infty$

Table 7.3

Pr	$-\frac{d\theta(0)}{d\eta}$	$0.508\left[\frac{20}{21}\frac{1}{Pr}+1\right]^{-1/4}(4Pr)^{1/4}$
0.01	0.0806	0.0725
0.03	0.136	0.1250
0.09	0.219	0.2133
0.5	0.442	0.4627
0.72	0.5045	0.5361
0.733	0.508	0.5399
1.0	0.5671	0.6078
1.5	0.6515	0.7031
2.0	0.7165	0.7751
3.5	0.8558	0.9253
5.0	0.954	1.0285
7.0	1.0542	1.1319
10	1.1649	1.2488
100	2.191	1.2665
1000	3.9660	4.0390

$$Nu_x\big|_{exact} = 0.503(Ra_x)^{1/4}, \quad Pr \to \infty \qquad (7.25b)$$

$$Nu_x\big|_{integral} = 0.508(Ra_x)^{1/4}, \quad Pr \to \infty \qquad (7.46b)$$

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NOTE:

(1) Error ranges from 1% for $Pr \to \infty$ to 14% for $Pr \to 0$

(2) We assumed $\delta = \delta_t$ ($Pr = 1$). Solution is reasonable for all Prandtl numbers

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