

$$\begin{cases} U_{tt} = c^2 U_{xx} - hu & 0 < x < l \quad t > 0 \\ U(0, t) = 0 \\ U(l, t) = 0 \\ U_t(x, 0) = 0 \\ U(x, 0) = \left[38 \sin\left(\frac{\pi x}{l}\right) \right] \end{cases}$$

از روش جداسازی متغیرها استفاده می‌کنیم: $U(x, t) = F(x)G(t)$ و به دست می‌آوریم:

$$U(x, t) = F(x)G(t) \rightarrow FG'' = c^2 F''G - hFG$$

$$\div FG \rightarrow \frac{G''}{G} = c^2 \frac{F''}{F} - h$$

$$\frac{F''}{F} = \frac{1}{c^2} \frac{G''}{G} + \frac{h}{c^2} = \begin{cases} k^2 \\ 0 \\ -k^2 \end{cases}$$

(1)

∴ $k^2 \rightarrow \frac{F''}{F} = k^2 \rightarrow F'' - k^2 F = 0 \rightarrow F(x) = A^{**} e^{-kx} + B^{**} e^{kx}$
 که در اینجا $k^2 > 0$ است.

$U(0, t) = 0 = F(0)G(t) = 0 \rightarrow F(0) = 0 \rightarrow A^{**} + B^{**} = 0$

$U(l, t) = 0 = F(l)G(t) = 0 \rightarrow F(l) = 0 \rightarrow A^{**} e^{kl} + B^{**} e^{-kl} = 0 \rightarrow A^{**} = B^{**}$

(2)

∴ $0 \rightarrow \frac{F''}{F} = 0 \rightarrow F'' = 0 \rightarrow F(x) = A^+ x + B^+$

$F(0) = 0 = A^+ \cdot 0 + B^+ \rightarrow A^+ = B^+$

$F(l) = 0 = A^+ l + A^+ \rightarrow A^+ = 0$

(3)

∴ $-k^2 \rightarrow \frac{F''}{F} = -k^2 \rightarrow F'' + k^2 F = 0$

$F(x) = A \cos kx + B \sin kx$

(4)

$$F(x) = 0 \rightarrow A = 0$$

$$F(x) = 0 \rightarrow B \sin kx = 0 \rightarrow \sin kx = 0 \rightarrow kx = n\pi, k = \frac{n\pi}{l}$$

$$\rightarrow F_n(x) = B_n \sin \frac{n\pi}{l} x$$

$$\frac{1}{c^2} \ddot{G} + \frac{h}{c^2} \dot{G} = -k^2 G \rightarrow \ddot{G} + \frac{h + k^2 c^2}{c^2} \dot{G} = 0$$

$$\lambda_n^2 = h + \left(\frac{n\pi c}{l}\right)^2$$

$$\ddot{G} + (h + k^2 c^2) G = 0 \rightarrow G_n(t) = C_n^* e^{\lambda_n t} + D_n^* \sin(\lambda_n t)$$

$$U_n(x,t) = G_n(t) F_n(x)$$

$$U_n(x,t) = (C_n^* e^{\lambda_n t} + D_n^* \sin \lambda_n t) \sin \frac{n\pi}{l} x$$

$$U(x,t) = \sum_{n=1}^{\infty} (C_n^* e^{\lambda_n t} + D_n^* \sin \lambda_n t) \sin \frac{n\pi}{l} x$$

$$U(x,0) = 3 \sin \frac{\pi x}{l} = \sum_{n=1}^{\infty} C_n^* \sin \frac{n\pi}{l} x$$

$$= 3 \sin \frac{\pi x}{l} = C_1^* \sin \frac{\pi x}{l} + C_2^* \sin \frac{2\pi}{l} x + \dots$$

$$C_1^* = 3$$

$$C_2^* = C_3^* = \dots = C_n^* = 0$$

$$U_+(x,0) = \sum_{n=1}^{\infty} (C_n^* e^{\lambda_n t} + D_n^* \sin \lambda_n t) \sin \frac{n\pi}{l} x \Big|_{t=0}$$

$$D_n^* = 0$$

$$u(x,t) = 3 \cos \lambda_1 t \sin \frac{n\pi}{l} x$$

$$= 3 \cos \left(h + \frac{\pi c_1^2}{l} \right) t \sin \frac{n\pi}{l} x$$

