CHAPTER 2

DIFFERENTIAL FORMULATION

OF THE BASIC LAWS

2.1 Introduction

- Solutions must satisfy 3 fundamental laws:

  conservation of mass
  conservation of momentum
  conservation of energy

- Differential formulation: application of basic laws to differential element

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2.2 Flow Generation

(i) Forced convection: by mechanical means(fan, blower, nozzle, jet, etc.)

(ii) Free (natural) convection: due to gravity and density change

2.3 Laminar vs. Turbulent Flow

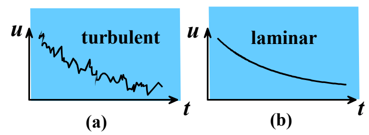

Fig. 2.1

Laminar: No random fluctuations
Turbulent: Random fluctuations

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Transition from laminar to turbulent:

Transition Reynolds number, depends on

- flow geometry
- surface roughness
- pressure gradient
- etc.

Flow over flat plate: $\approx$ 500,000

Flow through tubes: $\approx$ 2300

2.4 Conservation of Mass: The Continuity Equation

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2.4.1 Cartesian Coordinates

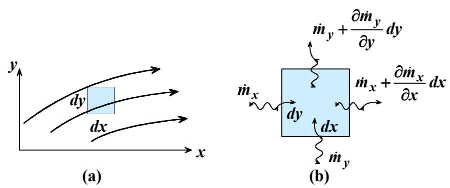

Fig. 2.2

Rate of mass added to element - Rate of mass remove from element = Rate of mass change within element (2.1)

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Assume continuum, use Fig. 2.2b, and (2.1)

$$\dot m_x + \dot m_y + \dot m_z - \left[\dot m_x + \frac{\partial \dot m_x}{\partial x} dx\right] - \left[\dot m_y + \frac{\partial \dot m_y}{\partial y} dy\right] + \left[\dot m_z + \frac{\partial \dot m_z}{\partial z} dz\right] = \frac{\partial m}{\partial t} \quad \text{(a)}$$

Express (a) in terms of density and velocity

$$\dot m = \rho V A \quad \text{(b)}$$

Apply (b) to element

$$\dot m_x = \rho u dydz \quad \text{(c)}$$

$$\dot m_y = \rho v dxdz \quad \text{(d)}$$

$$\dot m_z = \rho w dxdy \quad \text{(e)}$$

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Mass m of element

$$m = \rho\, dxdydz \quad \text{(f)}$$

(c)–(f) into (a)

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \quad \text{(2.2a)}$$

- **(2.2a) is the *continuity equation***

Alternate forms:

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x}\right] = 0 \quad \text{(2.2b)}$$

or

$$\frac{D\rho}{Dt} + \rho \nabla \cdot \vec{V} = 0 \quad \text{(2.2c)}$$

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or

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{V} = 0 \qquad (2.2d)$$

Special case: constant density (incompressible fluid)

(2.2c) becomes

$$\frac{D\rho}{Dt} = 0$$

$$\nabla \cdot \vec{V} = 0 \qquad (2.3)$$

2.4.2 Cylindrical Coordinates

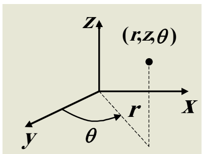


Fig. 2.3

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta) + \frac{\partial}{\partial z}(\rho v_z) = 0 \qquad (2.4)$$

2.4.3 Spherical Coordinates

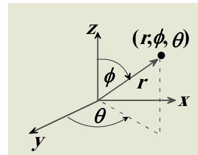
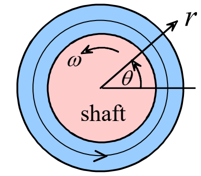


Fig. 2.4

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho r^2 v_r\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}(\rho v_\theta \sin\theta) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}\left(\rho v_\phi\right) = 0 \qquad (2.5)$$

Example 2.1: Fluid in Angular Motion

- **Shaft rotates inside tube**
- **Incompressible fluid**
- **No axial motion**
- **Give the continuity equation**



Solution

(1) Observations

- **Cylindrical coordinates**
- **No variation in axial and angular directions**
- **Incompressible fluid**

(2) Problem Definition. **Simplify the 3-D continuity**

(3) Solution Plan.

Apply the continuity equation in cylindrical coordinates

(4) Plan Execution

(i) Assumptions

- Incompressible
- No axial motion
- Shaft and tube are concentric (axisymmetric, no angular variation)

(ii) Analysis. Start with (2.4):

$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r v_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0 \qquad (2.4)$$

Simplify

Incompressible fluid: ρ is constant, $\frac{\partial \rho}{\partial t}=0$

No axial velocity: $v_z=0$

Axisymmetric: $\frac{\partial}{\partial \theta}=0$

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(2.4), gives $\frac{\partial}{\partial r}(r v_r)=0$ (a)

Integrate $r v_r = C$ (b)

C = constant of integration

Boundary condition: $v_r(r_o,\theta)=0$

Use (b) $C=0$ (c)

(b) gives $v_r = 0$ (d)

(iii) Checking

Dimensional check: Each term in (2.4) has units of density per unit time.

(5) Comments

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2.5 Conservation of Momentum: The Navier-Stokes Equation of Motion

2.5.1 Cartesian Coordinates

- Momentum is a vector quantity
- Newton's law of motion: 3 components
- Apply Newton's law to element, Fig. 2.5

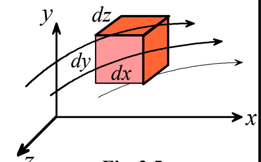

Fig. 2.5

$$\sum \delta \vec{F} = (\delta m)\vec{a} \qquad (a)$$

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$\vec{a}$ = acceleration of the element

$\delta\vec{F}$ = external force on element

δm = mass of the element

x-direction: $\sum \delta F_x = (\delta m) a_x$ (b)

Mass δm $\delta m = \rho dxdydz$ (c)

Total acceleration a_x

$$a_x = \frac{du}{dt} = \frac{Du}{Dt} = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t} \quad \text{(d)}$$

(c) and (d) into (b)

$$\sum \delta F_x = \rho \frac{Du}{Dt} dxdydz \quad \text{(e)}$$

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External *x*-forces:

(i) ***Body force*** (gravity)

(ii) ***Surface force***

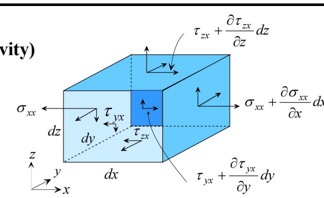

Fig. 2.6

Total forces

$$\sum \delta F_x = \sum \delta F_x)_{body} + \sum \delta F_x)_{surface} \quad \text{(f)}$$

Gravity force: $\sum \delta F_x)_{body} = \rho g_x dxdydz$ (g)

Surface forces:

σ_{xx} = normal stress on surface *dydz*

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τ_{yx} = shearing (tangential) stress on surface *dxdz*

τ_{zx} = shearing (tangential) stress on surface *dxdy*

Summing up *x*-forces, Fig. 2.6

$$\sum \delta F_x)_{surface} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dxdydz \quad \text{(h)}$$

Substituting (f), (g) and (h) into (e)

x-direction:

$$\rho \frac{Du}{Dt} = \rho g_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \quad \text{(2.6a)}$$

Similarly, for *y* and *z*-directions

y-direction:

$$\rho \frac{Dv}{Dt} = \rho g_y + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \quad \text{(2.6b)}$$

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z-direction:

$$\rho \frac{Dw}{Dt} = \rho g_z + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \quad (2.6c)$$

Unknowns in (2.6), 13:

$u, v, w, \rho, \sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \tau_{xy}, \tau_{yx}, \tau_{xz}, \tau_{zy}, \tau_{zx}, \tau_{yz}$

However

$$\tau_{xy} = \tau_{yx} \quad , \quad \tau_{xz} = \tau_{zx} \quad , \quad \tau_{yz} = \tau_{zy} \quad (i)$$

Reduce number of unknowns:

Use empirical relations called the *constitutive equations*

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \quad (2.7a)$$

$$\tau_{xz} = \tau_{zx} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \quad (2.7b)$$

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$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \quad (2.7c)$$

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x} - \frac{2}{3}\mu \nabla \cdot \vec{V} \quad (2.7d)$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y} - \frac{2}{3}\mu \nabla \cdot \vec{V} \quad (2.7e)$$

$$\sigma_{zz} = -p + 2\mu \frac{\partial w}{\partial z} - \frac{2}{3}\mu \nabla \cdot \vec{V} \quad (2.7f)$$

- Fluids obeying (2.7) are *Newtonian* fluids

  Substitute (2.7) into (2.6)

$$\rho \frac{Du}{Dt} = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla \cdot \vec{V}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \quad (2.8x)$$

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$$\rho \frac{Dv}{Dt} = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla \cdot \vec{V}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] \quad (2.8y)$$

$$\rho \frac{Dw}{Dt} = \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\nabla \cdot \vec{V}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \quad (2.8z)$$

NOTE:

- Eqs. (2.8) are the Navier-Stokes equations of motion
- Unknowns are 6: u, v, w, p, ρ, μ
- Restrictions: continuum and Newtonian fluid

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Vector form of (2.8x), (2.8y) and (2.8z)

$$\rho\frac{D\vec{V}}{Dt}=\rho\vec{g}-\nabla\vec{p}+\frac{4}{3}\nabla\left(\mu\nabla\cdot\vec{V}\right)+\nabla\left(\vec{V}\cdot\nabla\mu\right)-\vec{V}\nabla^2\mu+\nabla\mu\times\left(\nabla\times\vec{V}\right)-\left(\nabla\cdot\vec{V}\right)\nabla\mu-\nabla\times\left(\nabla\times\mu\vec{V}\right) \quad (2.8)$$

Simplified cases:

(i) Constant viscosity

$$\nabla\mu=0 \quad (j)$$

and

$$\nabla\times\left(\nabla\times\mu\vec{V}\right)=\nabla\left(\nabla\cdot\mu\vec{V}\right)-\nabla\cdot\nabla\mu\vec{V}=\mu\nabla\left(\nabla\cdot\vec{V}\right)-\mu\nabla^2\vec{V} \quad (k)$$

(j) and (k) into (2.8)

$$\rho\frac{D\vec{V}}{Dt}=\rho\vec{g}-\nabla\vec{p}+\frac{1}{3}\mu\nabla\left(\nabla\cdot\vec{V}\right)+\mu\nabla^2\vec{V} \quad (2.9)$$

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Eq. (2.9) is valid for: (1) continuum, (2) Newtonian (3) constant viscosity.

(ii) Constant viscosity and density

Continuity equation (2.3)

$$\nabla\cdot\vec{V}=0 \quad (2.3)$$

(2.3) into (2.9)

$$\rho\frac{D\vec{V}}{Dt}=\rho\vec{g}-\nabla\vec{p}+\mu\nabla^2\vec{V} \quad (2.10)$$

Eq. (2.10) is valid for: (1) continuum, (2) Newtonian (3) constant viscosity (4) constant density

The 3-components of (2.10):

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x-direction:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=\rho g_x-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right) \quad (2.10x)$$

z-direction:

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=\rho g_z-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \quad (2.10y)$$

y-direction:

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=\rho g_y-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \quad (2.10z)$$

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2.5.2 Cylindrical Coordinates

Assumptions: Continuum, (2) Newtonian fluid, (3) constant viscosity and (4) constant density.

***r*-direction:**

$$\rho\left(v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^{\ 2}}{r}+v_z\frac{\partial v_r}{\partial z}+\frac{\partial v_r}{\partial t}\right)=$$

$$\rho g_r-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right] \quad \text{(2.11r)}$$

θ -direction:

$$\rho\left(v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}-\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}+\frac{\partial v_\theta}{\partial t}\right)=$$

$$\rho g_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right] \quad \text{(2.11 }\theta\text{)}$$

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z-direction:

$$\rho\left(v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}+\frac{\partial v_z}{\partial t}\right)=$$

$$\rho g_z-\frac{\partial p}{\partial z}+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right] \quad \text{(2.11z)}$$

2.5.3 Spherical Coordinates

Assumptions: Continuum, (2) Newtonian fluid, (3) constant viscosity and (4) constant density.

***r*-direction:**

$$\rho\left(v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\theta^{\ 2}+v_\phi^{\ 2}}{r}+\frac{\partial v_r}{\partial t}\right)=$$

$$\rho g_r-\frac{\partial p}{\partial r}+\mu\left(\nabla^2 v_r-\frac{2}{r^2}v_r-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{2v_\theta\cot\theta}{r^2}-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right) \quad \text{(2.12r)}$$

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θ -direction:

$$\rho\left(v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}-\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{v_\phi^{\ 2}\cot\theta}{r}+\frac{\partial v_\theta}{\partial t}\right)=$$

$$\rho g_\theta-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left(\nabla^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right) \quad \text{(2.11 }\theta\text{)}$$

ϕ -direction:

$$\rho\left(v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi v_r}{r}+\frac{v_\theta v_\phi}{r}\cot\theta+\frac{\partial v_\theta}{\partial t}\right)=$$

$$\rho g_\phi-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}+$$

$$\mu\left(\nabla^2 v_\phi-\frac{v_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin^2\theta}\frac{\partial v_r}{\partial \phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right) \quad \text{(2.11}\phi\text{)}$$

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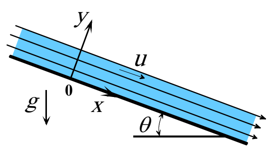
Where ∇^2 is

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \quad (2.13)$$

Example 2.2: Thin Liquid Film Flow Over an Inclined Surface

- Incompressible
- Parallel streamlines.
- Write the Navier-Stokes equations



(1) Observations

- Flow is due to gravity
- Parallel streamlines: $v = 0$
- Surface pressure is uniform (atmospheric)
- Cartesian geometry

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(2) Problem Definition.

Simplify the x and y components of the Navier-Stokes equations

(3) Solution Plan.

Start with the Navier-Stokes equations in Cartesian coordinates and simplify for this case

(4) Plan Execution

(i) Assumptions

- Newtonian
- steady state
- flow is in the x-direction only
- constant properties
- uniform ambient pressure
- parallel streamlines

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(ii) Analysis

Start with (2.10x) and (2.10y)

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad (2.10x)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \quad (2.10y)$$

Gravitational acceleration:

$$g_x = g\sin\theta \quad , \quad g_y = -g\cos\theta \quad (a)$$

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Simplifications:

Steady state: $\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} = 0$ (b)

Axial flow (*x*-direction only):

$$w = \frac{\partial}{\partial z} = 0 \quad \text{(c)}$$

Parallel streamlines: $v = 0$ (d)

(a)-(d) into (2.10x) and (2.10y)

$$\rho u \frac{\partial u}{\partial x} = \rho g \sin\theta - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{(e)}$$

and

$$0 = -\rho g \cos\theta - \frac{\partial p}{\partial y} \quad \text{(f)}$$

(f) is thy y-momentum equation

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Simplify (e) using continuity (2.3)

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{(g)}$$

(c) and (d) into (g)

$$\frac{\partial u}{\partial x} = 0 \quad \text{(h)}$$

(h) into (e)

$$\rho g \sin\theta - \frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{(i)}$$

Integrate (f)

$$p = -(\rho g \cos\theta)\, y + f(x) \quad \text{(j)}$$

$f(x)$ = "constant" of integration

At free surface, $y = H$, pressure is uniform equal to p_∞ .
Set $y = H$ in (j)

$$f(x) = p_\infty + \rho g H \cos\theta \quad \text{(k)}$$

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(k) into (j)

$$p = \rho g (H - y)\cos\theta + p_\infty \quad \text{(l)}$$

Different (k)

$$\frac{\partial p}{\partial x} = 0 \quad \text{(m)}$$

(m) into (i)

$$\rho g \sin\theta + \mu \frac{d^2 u}{d y^2} = 0 \quad \text{(n)}$$

This is the x-component

(iii) Checking

Dimensional check: Each term in (f) and (n) must have same units:

$\rho\, g \cos\theta$ = (kg/m^3)(*m/s^2*) = *kg/m^2-s^2*

$$\frac{\partial p}{\partial y} = \frac{\text{N/m}^2}{\text{m}} = \frac{\text{N}}{\text{m}^3} = \frac{\text{kg}-\text{m/s}^2}{\text{m}^3} = \text{kg/m}^2\text{-s}^2$$

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$\rho g \sin\theta$ = kg/m²-s²

$\mu \dfrac{d^2u}{dy^2}$ = (kg/m-s) $\dfrac{\text{m/s}}{\text{m}^2}$ = kg/m²-s²

Limiting check: For zero gravity fluid remains stationary. Set g = 0 in (n) gives

$$\frac{d^2u}{dy^2} = 0 \qquad \text{(o)}$$

Solution to (o): $u = 0$, ∴ fluid is stationary

(5) Comments

- Significant simplifications for: For 2-D incompressible, parallel flow
- The flow is 1-D since u depends on y only

2.6 Conservation of Energy: The Energy Equation

2.6.1 Cartesian Coordinates

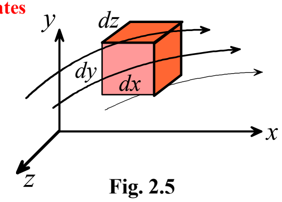

Fig. 2.5

Energy can not be created or destroyed

Apply to element $dxdydz$

A: Rate of change of internal and kinetic energy of element = B: Net rate of internal and kinetic energy transport by convection + C: Net rate of heat addition by conduction − D: Net rate of work done by element on surroundin gs (2.14)

- Express each term in (2.14) in terms of temperature (Appendix A)
- Explain physical significance of each term
- Result is called the energy equation
- Assumptions

- **Continuum**
- **Newtonian**
- **Negligible nuclear, electromagnetic and radiation energy**

(1) A = Rate of change of internal and kinetic energy of element

- **Internal energy of element depends on temperature (thermodynamic)**
- **Kinetic energy of element depends on velocity (flow field)**

$$\mathrm{A} = \frac{\partial}{\partial t}\left[\rho\left(\hat{u} + V^2/2\right)\right]dxdydz \qquad \text{(A-1)}$$

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(2) B = Net rate of internal and kinetic energy by convection

- **Internal energy convected through sides with mass flow. Depends on temperature**
- **Kinetic energy convected through sides of element with mass flow. Depends on velocity**

$$\mathrm{B} = -\left\{\nabla\cdot\left[\left(\hat{u} + V^2/2\right)\rho\vec{V}\right]\right\}dxdydz \qquad \text{(A-2)}$$

(3) C = Net rate of heat addition by conduction

- **Conduction at each surface depends on temperature gradient**
- **Apply Fourier's law (1.6)**

$$\mathrm{C} = -(\nabla\cdot\vec{q}'')\,dxdydz \qquad \text{(A-3)}$$

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(4) D = Net rate of work done by the element on the surroundings

Rate of work = force × velocity

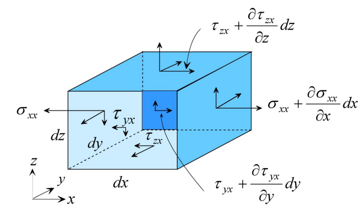

Fig. 2.6

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- 18 surface forces (Fig. 2.6)
- 3 body forces (gravity)
- Total 21 forces at 21 velocities

$$D = -\rho(\vec{V}\cdot\vec{g})dxdydz - \left[\frac{\partial}{\partial x}(u\sigma_{xx} + v\tau_{xy} + w\tau_{xz}) + \frac{\partial}{\partial y}(u\tau_{yx} + v\sigma_{yy} + w\tau_{yz}) + \frac{\partial}{\partial z}(u\tau_{zx} + v\tau_{zy} + w\sigma_{zz})\right]dxdydz \quad \text{(A-7)}$$

Substitute (A-1), (A-2), (A-3) and (A-7) into (2.14)

$$\frac{\partial}{\partial t}\left[\rho\left(\hat{u} + \frac{1}{2}V^2\right)\right] = -\nabla\cdot\left[\left(\hat{u} + \frac{1}{2}V^2\right)\rho\vec{V}\right] - \nabla\cdot\vec{q}'' + \rho(\vec{V}\cdot\vec{g}) + \left[\frac{\partial}{\partial x}(u\sigma_{xx} + v\tau_{xy} + w\tau_{xz}) + \frac{\partial}{\partial y}(u\tau_{yx} + v\sigma_{yy} + w\tau_{yz}) + \frac{\partial}{\partial z}(u\tau_{zx} + v\tau_{zy} + w\sigma_{zz})\right] \quad \text{(A-8)}$$

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Simplify using:

- Fourier's law (1.6)
- Continuity equation (2.2)
- Momentum equations (2.6)
- Constitutive equations (2.7)
- Thermodynamic relations for $\hat{u}$ and $\hat{h}$

$$\rho c_p \frac{DT}{Dt} = \nabla\cdot k\nabla T + \beta T\frac{Dp}{Dt} + \mu\Phi \quad \text{(2.15)}$$

where

β = coefficient of thermal expansion (compressibility)

- β is a property

$$\beta = -\frac{1}{\rho}\left[\frac{\partial\rho}{\partial T}\right]_p \quad \text{(2.16)}$$

Φ = dissipation function (energy due to friction)

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$$\Phi = 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right] + \left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)^2\right] - \frac{2}{3}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)^2 \quad \text{(2.17)}$$

- Φ is Important in high speed flow and for very viscous fluids

2.6.2 Simplified Form of the Energy Equation

(a) Cartesian Coordinates

- Use (2.15)
- Assumptions leading to (2.15):

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- Continuum
- Newtonian
- Negligible nuclear, electromagnetic and radiation energy transfer
- Special cases

(i) Incompressible fluid

$$\beta = 0$$

and

$$c_p = c_v = c$$

$$\rho c_p \frac{DT}{Dt} = \nabla \cdot k \nabla T + \mu \Phi \qquad (2.18)$$

(ii) Incompressible constant conductivity fluid

(2.18) is simplified further constant k:

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$$\rho c_p \frac{DT}{Dt} = k\nabla^2 T + \mu \Phi \qquad (2.19a)$$

or

$$\rho c_p \left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} \right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) + \mu \Phi \qquad (2.19b)$$

(iii) Ideal gas

$$\rho = \frac{p}{RT} \qquad (2.20)$$

(2.20) into (2.16)

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T} \right)_p = \frac{1}{\rho}\frac{p}{RT^2} = \frac{1}{T} \qquad (2.21)$$

(2.21) into (2.15)

$$\rho c_p \frac{DT}{Dt} = \nabla \cdot k \nabla T + \frac{Dp}{Dt} + \mu \Phi \qquad (2.22)$$

Using continuity (2.2c) and (2.20)

$$\rho c_v \frac{DT}{Dt} = \nabla \cdot k \nabla T - p\nabla \cdot \vec{V} + \mu \Phi \qquad (2.23)$$

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(b) Cylindrical Coordinates

Assume:

- Continuum
- Newtonian fluid
- Negligible nuclear, electromagnetic and radiation energy transfer
- Incompressible fluid
- Constant conductivity

$$\rho c_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r}\frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] + \mu \Phi \qquad (2.24)$$

where

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$$\Phi = 2\left(\frac{\partial v_r}{\partial r}\right)^2 + 2\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}\right)^2 + 2\left(\frac{\partial v_z}{\partial z}\right)^2 +$$

$$\left(\frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} + \frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)^2 + \left(\frac{1}{r}\frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z}\right)^2 + \left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)^2 \quad (2.25)$$

(c) Spherical Coordinates

Assume:

- **Continuum**
- **Newtonian fluid**
- **Negligible nuclear, electromagnetic and radiation energy transfer**
- **Incompressible fluid**
- **Constant conductivity**

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$$\rho c_p\left(\frac{\partial T}{\partial t} + v_r\frac{\partial T}{\partial r} + \frac{v_\phi}{r}\frac{\partial T}{\partial \phi} + \frac{v_\theta}{r\sin\phi}\frac{\partial T}{\partial \theta}\right) = \frac{k}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) +$$

$$k\left[\frac{1}{r^2\sin\phi\partial\phi}\left(\sin\phi\frac{\partial T}{\partial \phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 T}{\partial \theta^2}\right] + \mu\Phi \quad (2.26)$$

where

$$\Phi = 2\left[\left(\frac{\partial v_r}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r}\right)^2 + \left(\frac{1}{r\sin\phi}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} + \frac{v_\phi\cot\phi}{r}\right)^2\right] +$$

$$\left[r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \phi}\right]^2 +$$

$$\left[\frac{\sin\phi}{r}\frac{\partial}{\partial \phi}\left(\frac{v_\theta}{r\sin\phi}\right) + \frac{1}{r\sin\phi}\frac{\partial v_\theta}{\partial \theta}\right]^2 + \left[\frac{1}{r\sin\phi}\frac{\partial v_r}{\partial \phi} + r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right]^2 \quad (2.27)$$

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Example 2.3: Flow Between Parallel Plates

- **Axial flow with dissipation**
- **Assume:**
  - **Newtonian**
  - **Steady state**
  - **Constant density**
  - **Constant conductivity**
  - **Parallel streamlines**
- **Write the energy equation**

(1) Observations

- **Parallel streamlines:** $v = 0$
- **Incompressible, constant** k
- **Include dissipation**
- **Cartesian geometry**

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(2) Problem Definition

Determine the energy equation for parallel flow

(3) Solution Plan

Start with the energy equation for constant ρ and k in Cartesian coordinates and simplify

(4) Plan Execution

(i) Assumptions

- Newtonian
- Steady state
- Axial flow
- Constant ρ and k
- Negligible nuclear, electromagnetic and radiation energy transfer
- Parallel streamlines.

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(ii) Analysis. Start with energy equation (2.19b)

$$\rho c_P\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\mu\Phi \quad (2.19b)$$

where

$$\Phi=2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial z}\right)^2\right]+\left[\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)^2\right]-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2 \quad (2.17)$$

However

Steady state: $\frac{\partial T}{\partial t}=0$ (a)

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Axial flow: $w=\frac{\partial}{\partial z}=0$ (b)

Parallel flow: $v=0$ (c)

(a)-(c) into (2.19b) and (2.17)

$$\rho c_P u\frac{\partial T}{\partial x}=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right)+\mu\Phi \quad (d)$$

$$\Phi=2\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2-\frac{2}{3}\left(\frac{\partial u}{\partial x}\right)^2 \quad (e)$$

Further simplification: use continuity (2.3)

$$\nabla\cdot\vec{V}=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \quad (f)$$

(b) and (c) into (f) gives

$$\frac{\partial u}{\partial x}=0 \quad (g)$$

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(g) into (e)

$$\Phi = \left(\frac{\partial u}{\partial y}\right)^2 \qquad \text{(h)}$$

(h) into (d) gives the energy equation

$$\rho c_p u \frac{\partial T}{\partial x} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2 \qquad \text{(i)}$$

(iii) Checking

Dimensional check: Each term in (i) has the same units of W/m^3

Limiting check: For no fluid motion, energy equation reduces to pure conduction. Set $u = 0$ in (i)

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

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(5) Comments

- In energy equation (i), properties c_p, k, ρ and μ represent fluid nature
- Velocity u represents fluid motion
- Last term in (i) represents dissipation, making (i) non-linear

2.7 Solutions to the Temperature Distribution

Governing equations: continuity (2.2), momentum (2.8) and energy (2.15)

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TABLE 2.1

Basic law	No. of Equations	Unknowns							
Energy	1	T	u	v	w		ρ	μ	k
Continuity	1		u	v	w		ρ		
Momentum	3		u	v	w	p	ρ	μ	
Equation of State	1	T				p	ρ		
Viscosity relation $\mu = \mu(p,T)$	1	T				p		μ	
Conductivity relation $k = k(p,T)$	1	T				p			k

- Solution consideration: Table 2.1 Equation of state gives c_p and c_v and β

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(1) General case: variable properties

- 8 unknowns: $T, u, v, w, p, \rho, \mu, k$, 8 eqs. (yellow box)
- 8 eqs. solved simultaneously for 8 unknowns
- Velocity and temperature fields are *coupled.*

(2) Special case 1: constant k and μ

- 6 Unknowns: T, u, v, w, p, ρ, 6 eqs., see blue box
- 6 eqs. solved simultaneously for 6 unknowns

(3) Special case 2: constant k, μ and ρ

- 5 unknowns: T, u, v, w, p, 5 eqs., see red box
- However, 4 unknowns: u, v, w, p, 4 eqs., give flow field, see white box
- Velocity and temperature fields are *uncoupled*

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2.8 The Boussinesq Approximation

- Free convection is driven by density change
- Can't assume ρ = constant
- Alternate approach: the *Boussinesq approximation*
- Start with N-S equations for variable ρ

$$\rho \frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p + \frac{1}{3}\mu\nabla(\nabla \cdot \vec{V}) + \mu\nabla^2\vec{V} \quad (2.9)$$

- Assume:

(1) $\rho = \rho_\infty$ in inertia term

(2) $\rho = \rho_\infty$ in continuity, $\therefore \nabla . \vec{V} = 0$

(2.9) becomes

$$\rho_\infty \frac{D\vec{V}}{Dt} = \rho\vec{g} - \nabla p + \mu\nabla^2\vec{V} \quad (a)$$

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$(\,)_\infty$ Reference state (far away from object) where

$$\vec{V}_\infty = \text{uniform}, \quad \frac{D\vec{V}_\infty}{Dt} = \nabla^2\vec{V}_\infty = 0 \quad (b)$$

Apply(a) at infinity , use (b)

$$\rho_\infty\vec{g} - \nabla p_\infty = 0 \quad (c)$$

Subtract (c) from (a)

$$\rho \frac{D\vec{V}}{Dt} = (\rho - \rho_\infty)\vec{g} - \nabla(p - p_\infty) + \mu\nabla^2\vec{V} \quad (d)$$

(3) Express ($\rho - \rho_\infty$) in term of temperature difference.

Introduce β

$$\beta = -\frac{1}{\rho}\left[\frac{\partial\rho}{\partial T}\right]_p \quad (2.16)$$

Assume, for free convection $\rho(T, p) \approx \rho(T)$

$$\beta \approx -\frac{1}{\rho_\infty}\frac{d\rho}{dT} \quad (e)$$

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For small ΔT, is linear $\rho(T)$

$$\beta \approx -\frac{1}{\rho_\infty}\frac{\rho-\rho_\infty}{T-T_\infty} \qquad \text{(f)}$$

$\therefore$

$$\rho-\rho_\infty = -\beta\rho_\infty(T-T_\infty) \qquad \text{(2.28)}$$

Substitute (2.28) into (d)

$$\frac{D\vec{V}}{Dt} = -\beta\,\vec{g}(T-T_\infty) - \frac{1}{\rho_\infty}\nabla(p-p_\infty) + \nu\nabla^2\vec{V} \qquad \text{(2.29)}$$

- Simplification leading to (2.29) is called the Boussinesq approximation
- This eliminates density as a variable
- However, momentum and energy are coupled

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2.9 Boundary Conditions

(1) No-slip condition

At surface, $y = 0$

$$\vec{V}(x,0,z,t) = 0 \qquad \text{(2.30a)}$$

or

$$u(x,0,z,t) = v(x,0,z,t) = w(x,0,z,t) = 0 \qquad \text{(2.30b)}$$

(2) Free stream condition

- Far away from object, assume uniform velocity
- Example:

Uniform u at $y = \infty$:

$$u(x,\infty,z,t) = V_\infty \qquad \text{(2.31)}$$

Uniform temperature:

$$T(x,\infty,z,t) = T_\infty \qquad \text{(2.32)}$$

(3) Surface thermal conditions

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(i) Specified temperature

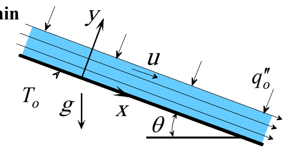
$$T(x,0,z,t) = T_s \qquad \text{(2.33)}$$

(ii) Specified heat flux

$$-k\frac{\partial T(x,0,z,t)}{\partial y} = q''_o \qquad \text{(2.34)}$$

Example 2.4: Heated Thin Liquid Film Flow Over an Inclined Surface

- Axial flow by gravity, thin film
- Uniform plate temperature T_o
- Uniform flux q''_o at free surface



- Write the velocity and thermal boundary conditions

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(1) Observations

- No slip condition at inclined plate
- Free surface is parallel to the inclined plate
- Specified temperature at plate
- Specified flux at free surface
- Cartesian geometry

(2) Problem Definition.

Write the boundary conditions at two surfaces for u, v and T

(3) Solution Plan

- Select an origin and coordinate axes
- Identify the physical flow and thermal conditions at the two surfaces
- Express conditions mathematically

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(4) Plan Execution

(i) Assumptions

- Constant film thickness
- Negligible shearing stress at free surface
- Newtonian fluid.

(ii) Analysis.

Origin and coordinates as shown

(1) No slip condition at the inclined surface

$$u(x,0) = 0 \tag{a}$$

$$v(x,0) = 0 \tag{b}$$

(2) Parallel streamlines

$$v(x,H) = 0 \tag{c}$$

(3) Negligible shear at free surface: for Newtonian fluid use (2.7a)

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$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \tag{2.7a}$$

Apply (2.7a) at the free surface, use (c)

$$\frac{\partial u(x,H)}{\partial y} = 0 \tag{d}$$

(4) Specified temperature at plate:

$$T(x,0) = T_o \tag{e}$$

(5) Specified heat flux at the free surface:

$$-k\frac{\partial T(x,0,z,t)}{\partial y} = -q''_o \tag{f}$$

(iii) Checking

Dimensional check: Each term of (f) has units of flux.

(5) Comments

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- Must select origin and coordinates
- Why negative heat flux in (f)?

2.10 Non-dimensional Form of the Governing Equations: Dynamic and Thermal Similarity Parameters

- Rewrite equations in dimensionless form to:
  - Identify governing *parameters*
  - Plan experiments
  - Present results
- Important factors in solutions
  - Geometry
  - Dependent variables: u, v, w, p, T
  - Independent variables: x, y, z, t

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- Constant quantities: $p_\infty, T_\infty, T_s, V_\infty, L, g$
- Fluid properties: c_p, k, β, μ, ρ
- Mapping results: dimensional vs. dimensionless

2.10.1 Dimensionless Variables

- To non-dimensionalize variables: use characteristic quantities $g, L, T_s, T_\infty, V_\infty$
- Define dimensionless variables

$$\vec{V}^* = \frac{\vec{V}}{V_\infty} \qquad p^* = \frac{(p - p_\infty)}{\rho_\infty V_\infty^2} \qquad T^* = \frac{(T - T_\infty)}{(T_s - T_\infty)}$$

$$x^* = \frac{x}{L} \qquad y^* = \frac{y}{L} \qquad z^* = \frac{z}{L} \qquad t^* = \frac{V_\infty}{L} t,$$

$$g^* = \frac{\vec{g}}{g} \tag{2.35}$$

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$$\therefore \quad \nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} = \frac{\partial}{L\partial x^*} + \frac{\partial}{L\partial y^*} + \frac{\partial}{L\partial z^*} = \frac{1}{L}\nabla^* \tag{2.36a}$$

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial}{L^2\partial x^{*2}} + \frac{\partial}{L^2\partial y^{*2}} + \frac{\partial}{L^2\partial z^{*2}} = \frac{1}{L^2}\nabla^{*2} \tag{2.36b}$$

$$\frac{D}{Dt} = \frac{D}{D(Lt^*/V_\infty)} = \frac{V_\infty}{L}\frac{D}{Dt^*} \tag{2.36c}$$

2.10.2 Dimensionless Form of Continuity

(2.35), (2.36) into (2.2c)

$$\frac{D\rho}{Dt^*} + \rho\nabla \cdot \vec{V}^* = 0 \tag{2.37}$$

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2.10.3 Dimensionless Form of the Navier-Stokes Equations of Motion

(2.35), (2.36) into (2.29)

$$\frac{D\vec{V}^*}{Dt^*} = -\frac{Gr}{Re^2}T^*\vec{g}^* - \nabla^* P^* + \frac{1}{Re}\nabla^{*2}\vec{V}^* \qquad (2.38)$$

Re and Gr are dimensionless parameters (numbers)

$$Gr \equiv \frac{\beta g(T_w - T_\infty)L^3}{\nu^2} \quad , \quad \textit{Reynolds number} \qquad (2.39)$$

$$Re \equiv \frac{\rho V_\infty L}{\mu} = \frac{V_\infty L}{\nu} \quad , \quad \textit{Grashof number} \qquad (2.40)$$

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2.10.4 Dimensionless Form of the Energy Equation

Two special cases:

(i) Incompressible, constant conductivity

(2.35), (2.36) into (2.19)

$$\frac{DT^*}{Dt^*} = \frac{1}{RePr}\nabla^{*2}T^* + \frac{E}{Re}\Phi^* \qquad (2.41a)$$

Pr and E are dimensionless parameters

$$Pr = \frac{c_p\mu}{k} = \frac{\mu/\rho}{k/\rho c_p} = \frac{\nu}{\propto} \quad , \quad \textit{Prandtl number} \qquad (2.42)$$

$$E = \frac{V_\infty{}^2}{c_p(T_s - T_\infty)} \quad , \quad \textit{Eckert number} \qquad (2.43)$$

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(2.35), (2.36) into (2.17) gives dimensionless dissipation function Φ^*

$$\Phi^* = 2\left[\left(\frac{\partial u^*}{\partial x^*}\right)^2 + \left(\frac{\partial v^*}{\partial y^*}\right)^2 + \cdots\right] \qquad (2.44)$$

(ii) Ideal gas, constant conductivity and viscosity

(2.35), (2.36) into (2.22)

$$\frac{DT^*}{Dt^*} = \frac{1}{RePr}\nabla^{*2}T^* + E\frac{Dp^*}{Dt^*} + \frac{E}{Re}\Phi^* \qquad (2.41b)$$

2.10.5 Significance of the Governing Parameters

Governing equations (2.37), (2.38), (2.41) are governed by 4 parameters: *Re, Pr, Gr and E:*

$$T^* = f(x^*, y^*, z^*.t^*;\ Re,\ Pr,\ Gr,\ E) \qquad (2.45)$$

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NOTE

- Significance of parameters
  - *Reynolds number*: *viscous effect*
  - *Prandtl number*: *property, heat transfer effect*
  - *Grashof number*: *buoyancy effect (free convection)*
  - *Eckert number*: *viscous dissipation: high speed flow and very viscous fluids*
- Dimensional form: solution depends on
  - 6 quantities: p_∞, T_∞, T_s, V_∞, L, g
  - 5 properties c_p, k, β, μ, and ρ *affect the solution*
- Dimensionless form: solution depends on
  - 4 parameters: *Re, Pr, Gr and E*

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- Special cases:
  - Negligible free convection: eliminate *Gr*
  - Negligible dissipation eliminate *E.*

$$T^* = f(x^*, y^*, z^*, t^*;\ Re,\ Pr) \qquad (2.46)$$

- Significance of (2.45) and (2.46):

Geometrically similar bodies have the same velocity and temperature solution if the parameters are the same

- Use (2.45) to:
  - Plan experiments
  - Carry out numerical computations
  - Organize presentation of results

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2.10.6 Heat Transfer Coefficient: The Nusselt Number

$$h = \frac{-k}{(T_s - T_\infty)} \frac{\partial T(x,0,z)}{\partial y} \qquad (1.10)$$

Express in dimensionless form: use (2.30)

$$\frac{hx}{k} = -x^* \frac{\partial T^*(x^*,0,z^*)}{\partial y^*} \qquad (2.47)$$

- *Local Nusselt number* Nu_x

$$Nu_x = \frac{hx}{k} \qquad (2.48)$$

- *Average Nusselt number* $\overline{Nu}$

$$\overline{Nu} = \frac{\overline{h}L}{k} \qquad (2.49)$$

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where

$$\bar{h} = \frac{1}{L}\int_0^L h(x)dx \qquad (2.50)$$

Recall

$$T^* = f(x^*, y^*, z^*. t^*;\ Re,\ Pr,\ Gr,\ E) \qquad (2.45)$$

Thus

$$Nu_x = f(x^*; Re, Pr, Gr, E) \qquad (2.51)$$

Special case: negligible buoyancy and viscous dissipation

$$Nu_x = f(x^*; Re, Pr) \qquad (2.52)$$

For free convection with negligible dissipation we obtain

$$Nu_x = f(x^*; Gr, Pr) \qquad (2.53)$$

For the average Nusselt number

$$\overline{Nu} = \frac{\bar{h}L}{k} = f(Re, Pr, Gr, E) \qquad (2.54)$$

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Example 2.5: Heat Transfer Coefficient for Flow over Cylinders

Two experiments, different cylinders, same fluid:

Experiment # 1	Experiment # 2
$D_1 = 3$ cm	$D_2 = 5$ cm
$V_1 = 15$ m/s	$V_2 = 98$ m/s
$\bar{h}_1 = 244$ W/m²-°C	$\bar{h}_2 = 144$ W/m²-°C

Compare results with correlation equation

$$\overline{Nu}_D = \frac{hD}{k} = C Re_D^{0.6} Pr^n \qquad (a)$$

Are experimental data accurate?

(1) Observations

- Compare data for $\bar{h}_1$ and $\bar{h}_2$ correlation (a)
- $\bar{h}$ appears in definition of $\overline{Nu}$

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- Fluid, C and n are unknown, (a) does not give $\bar{h}$
- Use (a) to determine ratio $\bar{h}_1 / \bar{h}_2$

(2) Problem Definition.

Determine $\bar{h}_1 / \bar{h}_2$ using data and correlation (a)

(3) Solution Plan.

Apply correlation (a) equation to determine $\bar{h}_1 / \bar{h}_2$ and compare experimental data

(4) Plan Execution

(i) Assumptions

- Correlation (a) is valid for both experiments
- Fluid properties are constant

(ii) Analysis

Use $Re_D = \frac{VD}{\nu}$ into (a)

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$$\frac{\bar{h}D}{k} = C\left(\frac{VD}{\nu}\right)^{0.6} Pr^n \quad \text{(b)}$$

Solve for $\bar{h}$

$$\bar{h} = \frac{C\ k\ V^{0.6} Pr^n}{\nu^{0.6} D^{0.4}} \quad \text{(c)}$$

Apply (c) to the two experiments

$$\bar{h}_1 = \frac{CkV_1^{\ 0.6} Pr^n}{\nu^{0.6} D_1^{\ 0.4}} \quad \text{(d)}$$

and

$$\bar{h}_2 = \frac{CkV_2^{\ 0.6} Pr^n}{\nu^{0.6} D_2^{\ 0.4}} \quad \text{(e)}$$

Take ratio of (d) and (e)

$$\frac{\bar{h}_1}{\bar{h}_2} = \left(\frac{V_1}{V_2}\right)^{0.6} \left(\frac{D_2}{D_1}\right)^{0.4} \quad \text{(f)}$$

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(iii) Computations

Substitute data for V_1, V_2, D_1 and D_2 into (f)

$$\frac{\bar{h}_1}{\bar{h}_2} = \left[\frac{15(\text{m/s})}{98\ \ (\text{m/s})}\right]^{0.6} \left[\frac{5(\text{cm})}{3(\text{cm})}\right]^{0.4} = 0.4 \quad \text{(g)}$$

Experimental data for ratio $\bar{h}_1 / \bar{h}_2$

$$\frac{\bar{h}_1}{\bar{h}_2} = \frac{244\left(\frac{\text{W}}{\text{m}^2\text{-}^\circ\text{C}}\right)}{144\left(\frac{\text{W}}{\text{m}^2\text{-}^\circ\text{C}}\right)} = 1.69 \quad \text{(h)}$$

The two results are not the same

Conclusion: Incorrect experimental data

(iv) Checking

Dimensional check: units of (f) are correct

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Limiting check: If $V_1 = V_2$ and $D_1 = D_2$, then $\bar{h}_1 = \bar{h}_2$.
This is confirmed by (f)

Qualitative check: If V is increased, $\bar{h}$ should increase. This is substantiated by (c).

(5) Comments

- Critical assumption: correlation (a) applies to both experiments
- Analysis suggests an error in the experimental data
- More conclusive check can be made if C, n and fluid are known

2.11 Scale Analysis

Procedure to obtain approximate results (order of magnitue) without solving equations

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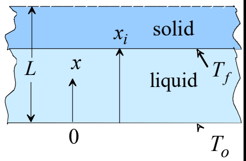
Example 2.6: Melting Time of Ice Sheet

- Ice sheet thickness L
- At freezing temperature T_f
- One side is at $T_o > T_f$
- Other side is insulated
- Conservation of energy at the melting front:



$$-k\frac{\partial T}{\partial x} = \rho \mathcal{L} \frac{dx_i}{dt} \quad \text{(a)}$$

x_i = melting front location

$\mathcal{L}$ = latent heat of fusion

- Use scale analysis to determine total melt time

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(1) Observations

- Entire sheet melts when $x_i = L$
- Largest temperature difference is $T_o - T_f$
- Time is in equation (a)
- Scaling of equation (a) should be useful

(2) Problem Definition

Determine the time $t = t_o$ when $x_i(t) = L$

(3) Solution Plan

Apply scale analysis to equation (a)

(4) Plan Execution

(i) Assumptions

- Sheet is perfectly insulated at $x = L$
- Liquid phase is stationary

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(ii) Analysis

Equation (a) is approximated by

$$-k\frac{\Delta T}{\Delta x} = \rho \mathcal{L} \frac{\Delta x_i}{\Delta t} \quad \text{(b)}$$

Select scales for variables in (a)

Scale for ΔT : $\Delta T \sim (T_o - T_f)$
Scale for Δx : $\Delta x \sim L$
Scale for Δx_i : $\Delta x_i \sim L$
Scale for Δt : $\Delta t \sim t_o$

Substitute into (a)

$$k\frac{(T_o - T_f)}{L} \sim \rho \mathcal{L} \frac{L}{t_o}$$

Solve for melt time t_o

$$t_o \sim \frac{\rho \mathcal{L} L^2}{k(T_o - T_f)} \quad \text{(c)}$$

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(iii) Checking

Dimensional check: Each term in (c) has units of time:

$$t_o = \frac{\rho\,(\text{kg/m}^3)\,\mathcal{L}(\text{J/kg})\,L^2(\text{m}^2)}{k(\text{W/m-}^\circ\text{C})(T_o - T_f)(^\circ\text{C})} = s$$

Limiting check:

(1) If *L is infinite, melt time is infinite. Set* $\mathcal{L} = \infty$ in (c) gives $t_o = \infty$

(2) If thickness is zero, melt time should vanish. Set *L = 0 in (c) gives* $t_o = 0$

Qualitative check:

Expect t_o to:

Directly proportional to mass, *L and L, and*

Inversely proportional to $k(T_o - T_f)$

This is confirmed by solution (c)

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(5) Comments

- t_o is estimated without solving governing equations
- Exact quasi-steady solution

$$t_o = \frac{\rho\,\mathcal{L}\,L^2}{2k(T_o - T_f)} \quad \text{(d)}$$

- Scaling answer is within a factor of 2

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