CHAPTER 4

BOUNDARY LAYER FLOW

APPLICATION TO EXTERNAL FLOW

4.1 Introduction

- Boundary layer concept (Prandtl 1904): Eliminate selected terms in the governing equations
- Two key questions

(1) What are the conditions under which terms in the governingequations can be dropped?

(2) What terms can be dropped?

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- Answer: By two approaches
  - Intuitive arguments
  - Scale analysis

4.2 The Boundary Layer Concept: Simplification of Governing Equations

4.2.1 Qualitative Description

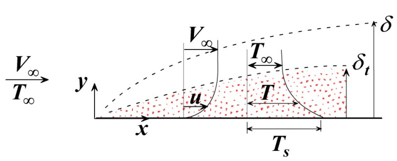

Fig. 4.1

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Under certain conditions the action of viscosity is confined to a thin region near the surface called the viscous or velocity boundary layer

Under certain conditions thermal interaction between moving fluid and a surface is confined to a thin region near the surface called the thermal or temperature boundary layer

- Conditions for viscous boundary layer:

(1) Slender body without flow separation
(2) High Reynolds number ($Re > 100$)

- Conditions for thermal boundary layer:

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(1) Slender body without flow separation
(2) High product of Reynolds and Prandtl numbers ($RePr > 100$)

$$\text{Peclet Number} = Pe = RePr = \frac{\rho V_\infty L}{\mu}\frac{c_p \mu}{k} = \frac{\rho c_p V_\infty L}{k} \qquad (4.1)$$

(1) Fluid velocity at surface vanishes
(2) Rapid changes across BL to V_∞
(3) Rapid changes temperature across BL from T_s to T_∞
(2) Boundary layers are thin:
For air at 10 m/s parallel to 1.0 m long plate, $\delta = 6$ mm at end
(3) Viscosity plays negligible role outside the viscous BL
(4) Boundary layers exist in both forced and free convection flows

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4.2.2 The Governing Equations

Simplified case:

Assumptions:

(1) steady state
(2) two-dimensional
(3) laminar
(4) constant properties
(5) no dissipation
(6) no gravity

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2.2)$$

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x-direction:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \qquad (2.10x)$$

y-direction:

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) \qquad (2.10y)$$

Energy:

$$\rho c_P\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \qquad (2.19)$$

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4.2.3 Mathematical Simplification

4.2.4 Simplification of the Momentum Equations

(i) Intuitive Arguments

Two viscous terms in (2.10x):

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

is one smaller than the other?

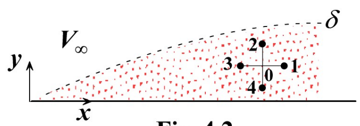

Fig. 4.2

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Insect dilemma: Too windy at position 0, where to go?
Move to position 4!
Conclusion:

Changes in *u* with respect to *y* are more pronounced than changes with respect to *x*

$$\frac{\partial^2 u}{\partial x^2} << \frac{\partial^2 u}{\partial y^2} \tag{4.2}$$

- **Neglect** $\frac{\partial^2 u}{\partial x^2}$ **in (2.10x)**

Pressure terms in (2.10x) and (2.10y):

- **Slender body**
- **Streamlines are nearly parallel**
- **Small vertical velocity**

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$$\frac{\partial p}{\partial y} \approx 0 \tag{4.3}$$

$\therefore$ **p depends on x only, i.e. $p \approx p(x)$**

$$\frac{\partial p}{\partial x} \approx \frac{dp}{dx} \approx \frac{dp_\infty}{dx} \tag{4.4}$$

(4.2) and (4.4) into (2.10x) gives:

Boundary layer x-momentum equation

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_\infty}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{4.5}$$

- **Continuity equation (2.2) and the x-momentum boundary layer equation (4.5) contain three unknowns: u, v, and p_∞**

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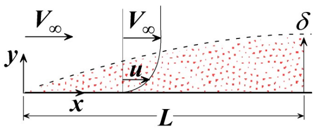
- p_∞ is pressure at edge of BL ($y = \delta$), obtained from solution of inviscid flow outside BL

(ii) Scale Analysis

- Use scaling to arrive at BL approximations.
- Assign a scale to each term in an equation

Slender body
Free stream velocity V_∞
Length L
BL thickness δ



Postulate:

Fig. 4.3

$$\frac{\delta}{L} << 1 \tag{4.6}$$

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If (4.6) is valid, we pose three questions:

(1) *What terms in the governing equations can be dropped?*

(2) *Is normal pressure gradient negligible compared to axial pressure gradient?*

(3) *Under what conditions is* (4.6) *valid?*

Assign scales:

$$u \sim V_\infty \tag{4.7a}$$

$$y \sim \delta \tag{4.7b}$$

$$x \sim L \tag{4.7c}$$

Apply (4.7) to continuity (2.2)

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

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Using (4.7)

$$\frac{v}{\delta} \sim \frac{V_\infty}{L}$$

Solving for v

$$v \sim V_\infty \frac{\delta}{L} \tag{4.7d}$$

Conclusion: $v << V_\infty$

Order of magnitude of inertia and viscous terms x-momentum equation (2.10x)

- First inertia term:

$$u\frac{\partial u}{\partial x} \sim V_\infty \frac{V_\infty}{L} \tag{a}$$

- Second inertial term:

$$v\frac{\partial u}{\partial y} \sim v\frac{V_\infty}{\delta}$$

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Use (4.7d)

$$v\frac{\partial u}{\partial y} \sim V_\infty \frac{V_\infty}{L} \quad \text{(b)}$$

Conclusion: 2 inertia terms are of the same order

Examine 2 viscous terms in (2.10x)

- First viscous term:

$$\frac{\partial^2 u}{\partial x^2} \sim \frac{V_\infty}{L^2} \quad \text{(c)}$$

- Second viscous term:

$$\frac{\partial^2 u}{\partial y^2} \sim \frac{V_\infty}{\delta^2} \quad \text{(d)}$$

Conclusion:

$$\frac{\partial^2 u}{\partial x^2} << \frac{\partial^2 u}{\partial y^2} \quad \text{(4.2)}$$

$\therefore$ Neglect $\partial^2 u / \partial x^2$ in (2.10x)

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Examine 2 viscous terms in (2.10y)

$$\frac{\partial^2 v}{\partial x^2} << \frac{\partial^2 v}{\partial y^2} \quad \text{(4.8)}$$

Simplify (2.10x) and (2.10y) Using (4.2) and (4.8)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \quad \text{(4.9x)}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial y^2} \quad \text{(4.9y)}$$

This answers first question

- Second question: pressure gradient

Scale $\frac{\partial p}{\partial x}$ and $\frac{\partial p}{\partial y}$

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Balance axial pressure with inertia in (4.9x)

$$\frac{\partial p}{\partial x} \sim \rho u \frac{\partial u}{\partial x} \quad \text{(e)}$$

Scale using (4.7)

$$\frac{\partial p}{\partial x} \sim \rho \frac{V_\infty^2}{L}$$

Balance pressure with inertial in (4.9y)

$$\frac{\partial p}{\partial y} \sim \rho \frac{V_\infty^2}{L}\frac{\delta}{L} \quad \text{(f)}$$

Compare (e) and (f) using (4.6)

$$\frac{\partial p}{\partial y} << \frac{\partial p}{\partial x} \quad \text{(4.10)}$$

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Since

$$p = p(x, y)$$

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

or

$$\frac{dp}{dx} = \frac{\partial p}{\partial x}\left[1 + \frac{(\partial p / \partial y)}{(\partial p / \partial x)}\frac{dy}{dx}\right] \tag{4.11}$$

Scale $\frac{dy}{dx}$

$$\frac{dy}{dx} \sim \frac{\delta}{L} \tag{g}$$

(e)-(g) into (4.11)

$$\frac{dp}{dx} = \frac{\partial p}{\partial x}\left[1 + (\delta / L)^2\right] \tag{h}$$

Invoke(4.6)

$$\frac{dp}{dx} \approx \frac{\partial p}{\partial x} \tag{i}$$

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Conclusion

Boundary layer pressure depends on x only. Variation with y is negligible

$\therefore$ Pressure $p(x)$ inside BL = pressure $p_\infty(x)$ at edge

$$p(x, y) \approx p_\infty(x) \tag{j}$$

$\therefore$

$$\frac{\partial p}{\partial x} \approx \frac{dp_\infty}{dx} \tag{4.12}$$

(4.12) into (4.9x)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{4.13}$$

(4.13) is x-momentum eq. for BL flow. Result is based on key assumption that $\delta / L << 1$.

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- Third question: condition for validity of (4.6)

$$\frac{\delta}{L} << 1 \tag{4.6}$$

Balance inertia with viscous force in (4.13)

Inertia:

$$u\frac{\partial u}{\partial x} \sim V_\infty \frac{V_\infty}{L} \tag{a}$$

Viscous:

$$\nu\frac{\partial^2 u}{\partial y^2} \sim \nu\frac{V_\infty}{\delta^2} \tag{b}$$

Equate

$$\frac{V_\infty^2}{L} \sim \nu\frac{V_\infty}{\delta^2}$$

Rearrange

$$\frac{\delta}{L} \sim \sqrt{\frac{\nu}{V_\infty L}} \tag{4.14a}$$

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or

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}} \qquad \text{(4.14b)}$$

where

$$Re_L = \frac{V_\infty L}{\nu} \qquad \text{(4.15)}$$

$$\frac{\delta}{L} << 1 \text{ when } \sqrt{Re_L} >> 1$$

Generalized (4.14)

$$\frac{\delta}{x} \sim \frac{1}{\sqrt{Re_x}} \qquad \text{(4.16)}$$

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4.2.5 Simplification of the Energy Equation

Simplify (2.19)

$$\rho c_P \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \qquad \text{(2.19)}$$

(i) Intuitive Arguments

Two conduction terms in (2.19):

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

is one smaller than the other?

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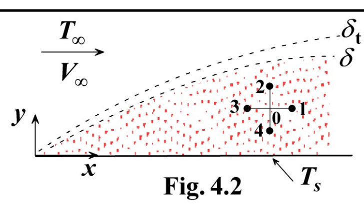

Fig. 4.2

Insect dilemma: Too hot at position 0, where to go?
Move to position 2!
Conclusion:

Changes in T with respect to y are more pronounced than with respect to x

$$\frac{\partial^2 T}{\partial x^2} << \frac{\partial^2 T}{\partial y^2} \qquad \text{(4.17)}$$

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• Neglect $\dfrac{\partial^2 T}{\partial x^2}$ in (2.19):

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2} \qquad (4.18)$$

(4.18) is the ***boundary layer energy equation.***

(ii) Scale Analysis

- Use scaling to arrive at BL approximations
- Assign a scale to each term in an equation

Slender body

Free stream velocity V_∞

Free stream temperature T_∞

Length L

BL thickness δ_t

Postulate:

$$\frac{\delta_t}{L} << 1 \qquad (4.19)$$

If (4.19) is valid, we pose two questions:

(1) *What terms in* (2.19) *can be dropped*?

(2) *Under what conditions is* (4.19) *valid*?

• Answer first question

Assign scales:

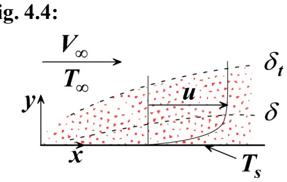
$$y \sim \delta_t \qquad (4.20)$$

$$\Delta T \sim T_s - T_\infty \qquad (4.21)$$

$$x \sim L \qquad (4.7b)$$

Scales for u and v depend on whether δ_t is larger or smaller than δ.

Two cases, Fig. 4.4:



case (1) : $\delta_t > \delta$

Fig. 4.4

Case (1): $\delta_t > \delta$

$$u \sim V_\infty \qquad (4.22)$$

Scaling of continuity:

$$v \sim V_\infty\frac{\delta_t}{L} \qquad (4.23)$$

Scales for convection terms in (2.19):

Use (4.7b) and (4.20-4.23)

$$u\frac{\partial T}{\partial x} \sim V_\infty \frac{\Delta T}{L} \quad \text{(a)}$$

and

$$v\frac{\partial T}{\partial y} \sim V_\infty \frac{\Delta T}{L} \quad \text{(b)}$$

Conclusion: the two terms are of the same order

Scale for conduction terms:

$$\frac{\partial^2 T}{\partial x^2} \sim \frac{\Delta T}{L^2} \quad \text{(c)}$$

and

$$\frac{\partial^2 T}{\partial y^2} \sim \frac{\Delta T}{\delta_t^2} \quad \text{(d)}$$

Compare (c) with (d), use : $\frac{\delta_t}{L} << 1$

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$$\therefore \quad \frac{\partial^2 T}{\partial x^2} << \frac{\partial^2 T}{\partial y^2} \quad \text{(e)}$$

$\therefore$ Energy equation simplifies to

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \quad (4.18)$$

Second question: *Under what conditions is* (4.19) *valid*?

$$\frac{\delta_t}{L} << 1 \quad (4.19)$$

Balance between convection and conduction:

$$u\frac{\partial T}{\partial x} \sim \alpha \frac{\partial^2 T}{\partial y^2}$$

Scaling

$$V_\infty \frac{\Delta T}{L} \sim \alpha \frac{\Delta T}{\delta_t^2}$$

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or

$$\frac{\delta_t}{L} \sim \sqrt{\frac{\alpha}{V_\infty L}}$$

or

$$\frac{\delta_t}{L} \sim \sqrt{\frac{k}{\rho c_p V_\infty L}}$$

or

$$\frac{\delta_t}{L} \sim \frac{1}{\sqrt{Pr Re_L}} \quad (4.24)$$

Conclusion:

$$\frac{\delta_t}{L} << 1 \quad \text{when} \quad \sqrt{Pr Re_L} >> 1 \quad (4.25)$$

Define Peclet number Pe

$$Pe = Pr Re_L \quad (4.26)$$

Example: For $Pe = 100$, $\frac{\delta_t}{L} \sim 0.1$

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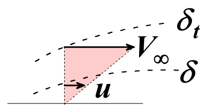
- When is $\delta_t > \delta$?
- Take ratio of (4.24) to (4.14b)

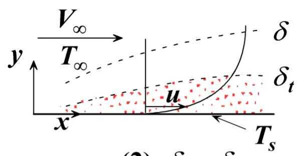
$$\frac{\delta_t}{\delta} \sim \frac{1}{\sqrt{Pr}} \tag{4.27}$$

$\therefore$ Criterion for : $\delta_t > \delta$

$$\delta_t > \delta \quad \text{when} \quad \sqrt{Pr} << 1 \tag{4.28}$$

Case (2): $\delta_t < \delta$ Fig. 4.4





case (2) : $\delta_t < \delta$

Fig. 4.4

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- u within the thermal boundary layer is smaller than free stream velocity
- Similarity of triangles

$$u \sim V_\infty \frac{\delta_t}{\delta} \tag{4.29}$$

Scaling of continuity

$$v \sim V_\infty \frac{\delta_t^2}{L\delta} \tag{4.30}$$

Use (4.29), (4.30) and follow procedure of case (1): conclusion:

(1) The two terms are of the same order

(2) Axial conduction is negligible compared to normal conduction

Second question: *Under what conditions is* (4.19) *valid*?

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Balance between convection and conduction:

$$u\frac{\partial T}{\partial x} \sim \alpha \frac{\partial^2 T}{\partial y^2}$$

Use (4.29) for u, scale each term

$$V_\infty \frac{\delta_t}{\delta}\frac{\Delta T}{L} \sim \alpha \frac{\Delta T}{\delta_t^2}$$

or

$$(\delta_t / L)^3 \sim \frac{k}{\rho c_p V_\infty L}\frac{\delta}{L} \tag{f}$$

However

$$\frac{\delta}{L} \sim \frac{1}{\sqrt{Re_L}} \tag{4.14b}$$

Substitute into (f)

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$$\frac{\delta_t}{L} \sim \frac{1}{Pr^{1/3}\sqrt{Re_L}} \qquad (4.31)$$

Conclusion:

$$\frac{\delta_t}{L} << 1 \quad \text{when} \quad Pr^{1/3}\sqrt{Re_L} >> 1 \qquad (4.32)$$

- **When is $\delta_t < \delta$?**
- **Take ratio of (4.31) to (4.14b)**

$$\frac{\delta_t}{\delta} \sim \frac{1}{Pr^{1/3}} \qquad (4.33)$$

$\therefore$ **Criterion for :** $\delta_t < \delta$

$$\delta_t < \delta \quad \text{when} \quad Pr^{1/3} >> 1 \qquad (4.34)$$

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4.3 Summary of Boundary Layer Equations for Steady Laminar Flow

Assumptions:

(1) Newtonian fluid

(2) two-dimensional

(3) negligible changes in kinetic and potential energy

(4) constant properties

- **Assumptions leading to boundary layer model**

(5) slender surface

(6) high Reynolds number ($Re > 100$)

(7) high Peclet number ($Pe > 100$)

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- **Introduce additional simplifications:**

(8) steady state

(9) laminar flow

(10) no dissipation ($\Phi = 0$)

(11) no gravity and

(12) no energy generation ($q''' = 0$)

Governing boundary layer equations:

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2.2)$$

***x*-Momentum:**

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (4.13)$$

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Energy:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad (4.18)$$

Note the following:

(1) Continuity is not simplified for boundary layer flow

(2) Pressure in (4.13) is obtained from inviscid solution outside BL. Thus (2.2) and (4.13) have two unknowns: u and v

(3) To include buoyancy, add $\rho\beta g(T - T_\infty)$ to right (4.13)

(4) Recall all assumptions leading the 3 equations

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4.4 Solutions: External Flow

- **Streamlined body in an infinite flow**
- **Examine thermal interaction**
- **Need temperature distribution T**
- **Temperature depends on velocity distribution**
- **For constant properties, velocity distribution is independent of temperature**

4.4.1 Laminar Boundary Layer Flow over Semi-infinite Flat Plate: Uniform Surface Temperature

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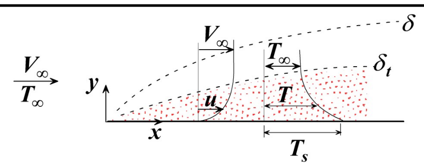

Fig. 4.5

- **Plate is at temperature T_s**
- **Upstream temperature is T_∞**
- **Upstream velocity uniform and parallel**
- **For assumptions listed in Section 4.3 the continuity, momentum and energy are given in (2.2), (4.13) and (4.18)**
- **Transition from laminar to turbulent at:**

$$Re_t = V_\infty x_t / \nu \approx 500{,}000$$

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(i) Velocity Distribution

Find:

- Velocity distribution
- Boundary layer thickness $\delta(x)$
- Wall shearing stress $\tau_o(x)$

(a) Governing equations and boundary conditions:

Continuity and x-momentum:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad (2.2)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{dp_\infty}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \qquad (4.13)$$

The velocity boundary conditions are:

$$u(x,0)=0 \qquad (4.35a)$$

$$v(x,0)=0 \qquad (4.35b)$$

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$$u(x,\infty)=V_\infty \qquad (4.35c)$$

$$u(0,y)=V_\infty \qquad (4.35d)$$

(b) Scale analysis: Find $\delta(x)$ and $\tau_o(x)$

Result of Section 4.2.4:

$$\frac{\delta}{x}\sim\frac{1}{\sqrt{Re_x}} \qquad (4.16)$$

Wall stress τ_o:

$$\tau_{xy}=\tau_{yx}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \qquad (2.7a)$$

At wall $y=0$, $v(x,0)=0$

$$\tau_o=\mu\frac{\partial u(x,0)}{\partial y} \qquad (4.36)$$

Scales for u and y:

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$$u\sim V_\infty \qquad (4.7a)$$

$$x\sim L \qquad (4.7c)$$

(4.36) is scaled using (4.7)

$$\tau_o\sim\mu\frac{V_\infty}{\delta} \qquad (a)$$

Use (4.16) for δ

$$\tau_o\sim\mu\frac{V_\infty}{x}\sqrt{Re_x} \qquad (b)$$

Friction coefficient C_f :

$$C_f=\frac{\tau_o}{(1/2)\rho V_\infty^2} \qquad (4.37a)$$

Use (b) for τ_o

$$C_f\sim\frac{1}{\sqrt{Re_x}} \qquad (4.37b)$$

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(c) Blasius solution: similarity method

- Solve (2.2) and (4.13) for the u and v
- Equations contain 3 unknowns: u, v, and p_∞
- Pressure is obtained from the inviscid solution outside BL

Inviscid solution:

- Uniform inviscid flow over slightly curved edge BL
- Neglect thickness δ
- Model: uniform flow over a flat plate of zero thickness
- Solution:

$$u = V_\infty, \quad v = 0, \quad p = p_\infty = \text{constant} \tag{4.38}$$

Thus the pressure gradient is

$$\frac{dp_\infty}{dx} = 0 \tag{4.39}$$

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(4.39) into (4.13)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{4.40}$$

- (4.40) is nonlinear
- Must be solved simultaneously with continuity (2.2)
- Solution was obtained by Blasius in 1908 using

similarity transformation:

Combine x and y ito a single variable $\eta(x, y)$

$$\eta(x, y) = y\sqrt{\frac{V_\infty}{\nu x}} \tag{4.41}$$

Postulate that $\dfrac{u}{V_\infty}$ depends on $\eta(x, y)$ only

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$$\frac{u}{V_\infty} = \frac{df}{d\eta} \tag{4.42}$$

$f = f(\eta)$ to be determined

NOTE:

(1) Including $\sqrt{V_\infty / \nu}$ in definition of η , is for convenience only

(1) $\eta(x, y)$ in (4.41) is arrived at by formal procedure

Continuity (2.2) gives v:

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}$$

Multiplying by dy, integrate

$$v = -\int \frac{\partial u}{\partial x} dy \tag{a}$$

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Use (4.41) and (4.42) to express dy and $\partial u / \partial x$ in terms of the variable η

$$dy = \sqrt{\frac{V_\infty}{\nu x}}\, d\eta \qquad \text{(b)}$$

Chain rule:

$$\frac{\partial u}{\partial x} = \frac{du}{d\eta}\frac{d\eta}{dx}$$

Use (4.41) and (4.42) into above

$$\frac{\partial u}{\partial x} = -\frac{V_\infty}{2x}\frac{d^2 f}{d\eta^2}\eta \qquad \text{(c)}$$

(b) and (c) into (a)

$$\frac{v}{V_\infty} = \frac{1}{2}\sqrt{\frac{\nu}{V_\infty x}} \int \frac{d^2 f}{d\eta^2}\eta\, d\eta$$

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Integration by parts gives

$$\frac{v}{V_\infty} = \frac{1}{2}\sqrt{\frac{\nu}{V_\infty x}}\left(\eta \frac{df}{d\eta} - f\right) \qquad \text{(4.43)}$$

- Need function $f(\eta)$, use momentum equation

First determine $\partial u / \partial y$ and $\partial^2 u / \partial y^2$

$$\frac{\partial u}{\partial y} = \frac{du}{d\eta}\frac{d\eta}{dy} = V_\infty \frac{d^2 f}{d\eta^2}\sqrt{\frac{V_\infty}{\nu x}} \qquad \text{(d)}$$

$$\frac{\partial^2 u}{\partial y^2} = V_\infty \frac{d^3 f}{d\eta^3}\frac{V_\infty}{\nu x} \qquad \text{(e)}$$

(4.42), (4.43) and (c)-(e) into (4.40)

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$$2\frac{d^3 f}{d\eta^3} + f(\eta)\frac{d^2 f}{d\eta^2} = 0 \qquad \text{(4.44)}$$

Partial differential equations are transformed into an ordinary differential equation

NOTE: x and y are eliminated in (4.44)

- Transformation of boundary conditions

$$\frac{df(\infty)}{d\eta} = 1 \qquad \text{(4.45a)}$$

$$f(0) = 0 \qquad \text{(4.45b)}$$

$$\frac{df(0)}{d\eta} = 0 \qquad \text{(4.45c)}$$

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$$\frac{df(\infty)}{d\eta} = 1 \qquad (4.45d)$$

Equations (4.44) is third order. How many boundary conditions?

- **Difficulty: (4.44) is nonlinear**
- **Solution by power series (Blasius)**
- **Result: Table 4.1**

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Table 4.1 Blasius solution [1]

$\eta = y\sqrt{\frac{V_\infty}{\nu x}}$	f	$\frac{df}{d\eta} = \frac{u}{V_\infty}$	$\frac{d^2 f}{d\eta^2}$
0.0	0.0	0.0	0.33206
0.4	0.02656	0.13277	0.33147
0.8	0.10611	0.26471	0.32739
2.4	0.92230	o.72899	0.22809
2.8	1.23099	0.81152	0.18401
3.2	1.56911	0.87609	0.13913
3.6	1.92954	0.92333	0.09809
4.0	2.30576	0.95552	0.06424
4.4	2.69238	0.97587	0.03897
4.8	3.08534	0.98779	0.02187
5.0	3.28329	0.99155	0.01591
5.2	3.48189	0.99425	0.01134
5.4	3.68094	0.99616	0.00793
5.6	3.88031	0.99748	0.00543

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- **Find $\delta(x)$ wall stress $\tau_o(x)$**
- **Define δ as the distance y from the plate where u/V_∞ = 0.994, Table 4.1 gives**

$$\delta = 5.2\sqrt{\frac{\nu x}{V_\infty}}$$

or

$$\frac{\delta}{x} = \frac{5.2}{\sqrt{Re_x}} \qquad (4.46)$$

Scaling result:

$$\frac{\delta}{x} \sim \frac{1}{\sqrt{Re_x}} \qquad (4.16)$$

- **Wall stress τ_o: use**

$$\tau_o = \mu\frac{\partial u(x,0)}{\partial y} \qquad (4.36)$$

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(d) into (4.36), use Table 4.1

$$\tau_o = \mu V_\infty \sqrt{\frac{V_\infty}{\nu x}} \frac{d^2 f(0)}{d\eta^2} = 0.33206\, \mu V_\infty \sqrt{\frac{V_\infty}{\nu x}} \tag{4.47}$$

Friction coefficient C_f : (4.47) into (4.37a)

$$C_f = \frac{0.664}{\sqrt{Re_x}} \tag{4.48}$$

Scaling result:

$$C_f \sim \frac{1}{\sqrt{Re_x}} \tag{4.37b}$$

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(ii) Temperature Distribution

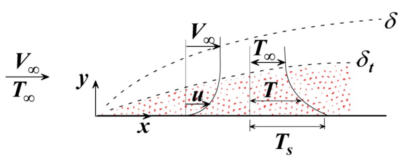

Fig. 4.5

- Isothermal semi-infinite plate
- Determine: δ_t, $h(x)$ and Nu_x
- Need temperature distribution

(a) Governing equation and boundary conditions

Assumption: Listed in Section 4.3

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Energy equation

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{4.18}$$

The boundary condition are:

$$T(x,0) = T_s \tag{4.49a}$$

$$T(x,\infty) = T_\infty \tag{4.49b}$$

$$T(0,y) = T_\infty \tag{4.49c}$$

(b) Scale analysis: δ_t, $h(x)$ and Nu_x

From Section 4.2.5: Set $L = x$ (4.24) and (4.31)

Case (1): $\delta_t > \delta$ (Pr<<1)

$$\frac{\delta_t}{x} \sim \frac{1}{\sqrt{PrRe_x}} \tag{4.50}$$

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Case (2): $\delta_t < \delta$ (Pr>>1)

$$\frac{\delta_t}{x} \sim \frac{1}{Pr^{1/3}\sqrt{Re_x}} \qquad (4.51)$$

Heat transfer coefficient $h(x)$

$$h = -k\frac{\dfrac{\partial T(x,0)}{\partial y}}{T_s - T_\infty} \qquad (1.10)$$

Use scales of (4.20) and (4.21) into above

$$h \sim \frac{k}{\delta_t} \qquad (4.52)$$

Where δ_t **is given by (4.50) and (4.51).**

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Case (1): $\delta_t > \delta$ (Pr<<1), (4.50) into (4.52)

$$h \sim \frac{x}{k}\sqrt{PrRe_x} \quad , \quad \text{for } Pr<<1 \qquad (4.53)$$

Local Nusselt number Nu_x

$$Nu_x = \frac{hx}{k} \qquad (4.54)$$

(4.53) into (4.54)

$$Nu_x \sim \sqrt{PrRe_x} \quad , \quad \text{for } Pr<<1 \qquad (4.55)$$

Case (2): $\delta_t << \delta$ (Pr>>1). Substituting (4.51) into (4.52)

$$h \sim \frac{k}{x}Pr^{1/3}\sqrt{Re_x} \quad , \quad \text{for } Pr>>1 \qquad (4.56)$$

Nusselt number:

$$Nu_x \sim Pr^{1/3}\sqrt{Re_x} \quad , \quad \text{for } Pr>>1 \qquad (4.57)$$

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(c) Pohlhausen's solution: $T(x,y)$, δ_t , $h(x)$, Nu_x

- **Energy equation (4.18) is solved analytically**
- **Solution by Pohlhausen (1921) using similarity transformation**
- **Defined** θ

$$\theta = \frac{T - T_s}{T_\infty - T_s} \qquad (4.58)$$

(4.58) into (4.18)

$$u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} = \alpha\frac{\partial^2\theta}{\partial y^2} \qquad (4.59)$$

B.C.

$$\theta(x,0) = 0 \qquad (4.60a)$$

$$\theta(x,0) = 1 \qquad (4.60b)$$

$$\theta(x,0) = 1 \qquad (4.60c)$$

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• Solve (4.59) and (4.60) using similarity

• Introduce transformation variable η

$$\eta(x,y) = y\sqrt{\frac{V_\infty}{\nu x}} \tag{4.41}$$

Assume

$$\theta(x,y) = \theta(\eta)$$

• Blasius solution gives u and v

$$\frac{u}{V_\infty} = \frac{df}{d\eta} \tag{4.42}$$

$$\frac{v}{V_\infty} = \frac{1}{2}\sqrt{\frac{\nu}{V_\infty x}}\left(\eta\frac{df}{d\eta} - f\right) \tag{4.43}$$

(4.41)-(4.43) into (4.59) and noting that

$$\frac{\partial\theta}{\partial x} = \frac{d\theta}{d\eta}\frac{\partial\eta}{\partial x} = -\frac{\eta}{2x}\frac{d\theta}{d\eta}$$

$$\frac{\partial\theta}{\partial y} = \frac{d\theta}{d\eta}\frac{\partial\eta}{\partial y} = \sqrt{\frac{V_\infty}{\nu x}}\frac{d\theta}{d\eta}$$

$$\frac{\partial^2\theta}{\partial y^2} = \frac{V_\infty}{\nu x}\frac{d^2\theta}{d\eta^2}$$

(4.59) becomes

$$\frac{d^2\theta}{d\eta^2} + \frac{Pr}{2} f(\eta)\frac{d\theta}{d\eta} = 0 \tag{4.61}$$

Result:

Partial differential equation is transformed into an ordinary differential equation

NOTE:

(1) One parameter: Prandtl number Pr

(2) (4.61) is linear, 2nd order ordinary D.E.

(3) $f(\eta)$ in (4.61) represents the effect motion

Transformation of B.C.:

$$\theta(\infty) = 1 \tag{4.62a}$$

$$\theta(0) = 0 \tag{4.62b}$$

$$\theta(\infty) = 1 \tag{4.62c}$$

Solution: Separate variables, integrate twice, use B.C. (4.62) (Details in Appendix)

$$\theta(\eta) = 1 - \frac{\int_\eta^\infty \left[\frac{d^2 f)}{d\eta^2}\right]^{Pr} d\eta}{\int_0^\infty \left[\frac{d^2 f}{d\eta^2}\right]^{Pr} d\eta} \tag{4.63}$$

Surface temperature gradient:

$$\frac{d\theta(0)}{d\eta} = \frac{[0.332]^{Pr}}{\int_{\eta}^{\infty}\left[\frac{d^2 f}{d\eta^2}\right]^{Pr} d\eta} \qquad (4.64)$$

- **Integrals are evaluated numerically**
- $\frac{d^2 f}{d\eta^2}$ **is obtained from Blasius solution**
- **Results are presented graphically in Fig. 4.6**

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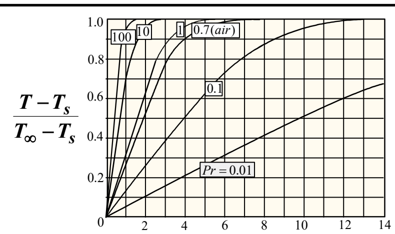

$$\eta = y\sqrt{\frac{V_\infty}{\nu x}}$$

Fig. 4.6 Pohlhausen's solution

- **Determine:** δ_t, $h(x)$ **and** Nu_x
- **Fig. 4.6 gives** δ_t **. At** $y = \delta_t$, $T \approx T_\infty$, **or**

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$$\theta = \frac{T - T_s}{T_\infty - T_s} \approx 1 \;, \quad \text{at} \quad y = \delta_t \qquad (4.65)$$

- **Fig. 4.6 shows that** $\delta_t(x)$ **depends on** Pr
- **Local heat transfer coefficient** $h(x)$**:** **use (1.10)**

$$h = -k\,\frac{\dfrac{\partial T(x,0)}{\partial y}}{T_s - T_\infty} \qquad (1.10)$$

where

$$\frac{\partial T(x,0)}{\partial y} = \frac{dT}{d\theta}\frac{d\theta(0)}{d\eta}\frac{\partial \eta}{\partial y}$$

Use (4.41) and (4.58) into above

$$\frac{\partial T(x,0)}{\partial y} = (T_\infty - T_s)\sqrt{\frac{V_\infty}{\nu x}}\,\frac{d\theta(0)}{d\eta}$$

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Substitute into (1.10)

$$h(x) = k\sqrt{\frac{V_\infty}{\nu x}}\frac{d\theta(0)}{d\eta} \tag{4.66}$$

- **Average heat transfer coefficient:**

$$\bar{h} = \frac{1}{L}\int_0^L h(x)dx \tag{2.50}$$

Use (4.66) and integrate

$$\bar{h} = 2\frac{k}{L}\sqrt{Re_L}\frac{d\theta(0)}{d\eta} \tag{4.67}$$

- **Local Nusselt number:** (4.66) into (4.54)

$$Nu_x = \frac{d\theta(0)}{d\eta}\sqrt{Re_x} \tag{4.68}$$

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- **Average Nusselt number:**

$$\overline{Nu_L} = 2\frac{d\theta(0)}{d\eta}\sqrt{Re_L} \tag{4.69}$$

- **Total heat transfer rate q_T:**

Plate length L and width W. Apply Newton's law

$$q_T = \int_0^L h(x)(T_s - T_\infty)W\,dx =$$

$$(T_s - T_\infty)W\int_0^L h(x)dx = (T_s - T_\infty)WL\bar{h}$$

or

$$q_T = (T_s - T_\infty)A\bar{h} \tag{4.70}$$

Heat transfer coefficient and Nusselt number depend on surface temperature gradient $\dfrac{d\theta(0)}{d\eta}$

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- $\dfrac{d\theta(0)}{d\eta}$ depends on Pr
- It is determined from (4.64)
- Values in Table 4.2
- Approximate values of $\dfrac{d\theta(0)}{d\eta}$ are given by:

Table 4.2

Pr	$\dfrac{d\theta(0)}{d\eta}$
0.001	0.0173
0.01	0.0516
0.1	0.140
0.5	0.259
0.7	0.292
1.0	0.332
7.0	0.645
10.0	0.730
15.0	0.835
50	1.247
100	1.572
1000	3.387

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$$\frac{d\theta(0)}{d\eta} = 0.564\, Pr^{1/2} \quad , \quad Pr < 0.05 \qquad (4.71a)$$

$$\frac{d\theta(0)}{d\eta} = 0.332\, Pr^{1/3} \quad , \quad 0.6 < Pr < 10 \qquad (4.71b)$$

$$\frac{d\theta(0)}{d\eta} = 0.339\, Pr^{1/3} \quad , \quad Pr > 10 \qquad (4.71c)$$

- **Compare with scaling:**
- **Two cases: $Pr << 1$ and $Pr >> 1$**

Combine (4.71a) and (4.71c) with (4.68)

$$Nu_x = 0.564\, Pr^{1/2}\sqrt{Re_x}, \ \text{for} \ Pr < 0.05 \qquad (4.72a)$$

$$Nu_x = 0.339\, Pr^{1/3}\sqrt{Re_x}, \ \text{for} \ Pr > 10 \qquad (4.72c)$$

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Scaling results:

$$Nu_x \sim \sqrt{PrRe_x} \quad , \quad \text{for} \ Pr << 1 \qquad (4.55)$$

$$Nu_x \sim Pr^{1/3}\sqrt{Re_x} \quad , \quad \text{for} \ Pr >> 1 \qquad (4.57)$$

- **Fluid properties: Evaluated at the film temperature T_f**

$$T_f = \frac{T_s + T_\infty}{2} \qquad (4.73)$$

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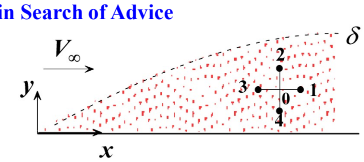
4.4.2 Applications: Blasius Solution, Pohlhausen's Solutions and Scaling

- **Three examples**

Example 4.1: Insect in Search of Advice

- **Air at 30°C,**
  $V_\infty = 4$ m/s
- **Insect at 0**



- **Determine velocity u at locations 0, 1, 2, 3, 4.**
- **Is insect inside BL?**

(1) Observations.

- **External forced convection boundary layer problem**

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- Changes in velocity between 1 and 3 should be small compared to those between 2 and 4
- Location 4 should have the lowest velocity
- If the flow is laminar Blasius applies
- The flow is laminar if Reynolds number is less than 500,000

(2) Problem Definition. Determine u at the five locations

(3) Solution Plan.

- Check the Reynolds number for BL approximations and if the flow is laminar
- If laminar, use Blasius solution, Table 4.1, to determine u and δ

(4) Plan Execution

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(i) Assumptions. All assumptions leading to Blasius solution: These are:

- Newtonian fluid
- steady state
- constant properties
- two-dimensional
- laminar flow ($Re_x < 5\times10^5$)
- viscous boundary layer flow ($Re_x > 100$)
- (7) uniform upstream velocity
- flat plate
- negligible changes in kinetic and potential energy
- no buoyancy ($\beta = 0$ or $g = 0$)

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(ii) Analysis

$$Re_x = \frac{V_\infty x}{\nu} \qquad \text{(a)}$$

V_∞ = upstream velocity = 4 m/s

ν = kinematic viscosity = 16.01 $\times 10^{-6}$ m² /s

Transition Reynolds number:

$$Re_{x_t} = 5\times10^5 \qquad \text{(b)}$$

- Laminar flow if $Re_x < Re_{x_t}$
- Viscous BL approximations are valid for

$$Re_x > 100 \qquad \text{(c)}$$

At x = 151 mm:

$$Re_x = \frac{4(\text{m/s})0.151(\text{m})}{16.01\times10^{-6}(\text{m}^2/\text{s})} = 37{,}726$$

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$\therefore$ **BL flow is laminar. Use Blasius solution**

Determine

$$\eta = y\sqrt{\frac{V_\infty}{\nu x}} \qquad \text{(d)}$$

$$\frac{\delta}{x} = \frac{5.2}{\sqrt{Re_x}} \qquad (4.46)$$

(iii) Computations.

- **Calculate η at each location, use Table 4.1 to find u/V_∞.**

Results:

location	x (m)	y (m)	η	u/V_∞	u(m/s)
0	0.150	0.002	2.581	0.766	3.064
1	0.151	0.002	2.573	0.765	3.06
2	0.150	0.003	3.872	0.945	3.78
3	0.149	0.002	2.59	0.768	3.072
4	0.150	0.001	1.291	0.422	1.688

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- **Use (4.46) to determine δ at $x = 0.151$ m and $Re_x = 37{,}726$**

$$\delta = \frac{5.2}{\sqrt{Re_x}}x = \frac{5.2}{\sqrt{37{,}726}}0.151(\text{m}) = 0.004\,\text{m} = 4\,\text{mm}$$

Thus the insect is within the boundary layer

(iv) Checking. ***Dimensional check*:**

Equations (a) and (d) are dimensionally correct

***Qualitative check*: u at the five locations follow expected behavior**

(5) Comments.

- **The insect should move to location 4**
- **Changes in u with respect to x are minor**
- **Changes in u with respect to y are significant**

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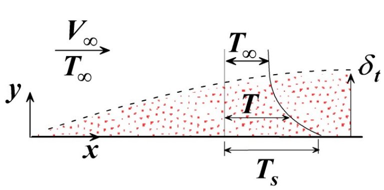
- **What is important for the insect is the magnitude of the velocity vector $V = (u^2 + v^2)^{1/2}$ and not u. However, since $v << u$ in boundary layer flow, using u as a measure of total velocity is reasonable**

Example 7.2: Laminar Convection over a Flat Plate

- **Water**
- **$V_\infty = 0.25$ m/s**
- **$T_\infty = 35°C$**
- **$T_s = 85°C$**
- **$L = 75$ cm**



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[a] Find equation for $\delta_t(x)$

[b] Determine h at x = 7.5 cm and 75 cm

[c] Determine q_T for a plate 50 cm wide

[d] Can Pohlhausen's solution be used to q'' at the trailing end of the plate if its length is doubled?

(1) Observations

- External forced convection over a flat plate
- $\delta_t(x)$ increases with x
- Newton's law of cooling gives q'' and q_T
- $h(x)$ decreases with x
- Pohlhausen's solution is applies laminar flow and all other assumptions made
- Doubling the length doubles the Reynolds number

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(2) Problem Definition. Determine temperature distribution

(3) Solution Plan

- Compute the Reynolds and Peclet numbers to establish if this is a laminar boundary layer problem
- Use Pohlhausen's solution to determine δ_t , $h(x)$, q'' and q_T

(4) Plan Execution

(i) Assumptions. All assumptions leading to Blasius solution: These are:

- Newtonian fluid
- two-dimensional
- negligible changes in kinetic and potential energy
- constant properties
- boundary layer flow

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- steady state
- laminar flow
- no dissipation
- no gravity
- no energy generation
- flat plate
- negligible plate thickness
- uniform upstream velocity V_∞
- uniform upstream temperature T_∞
- uniform surface temperature T_s
- no radiation

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(ii) Analysis and Computations

- **Are BL approximations valid? Calculate the Reynolds and Peclet. Condition:**

$$Re_x > 100 \quad \text{and} \quad Pe = Re_x Pr > 100 \qquad \text{(a)}$$

$$Re_x = \frac{V_\infty x}{\nu}$$

Transition Reynolds number: Re_t

$$Re_x = 5 \times 10^5 \qquad \text{(b)}$$

Properties at T_f

$$T_f = (T_s + T_\infty)/2 \qquad \text{(c)}$$

$T_s = 85^o\text{C}$

$T_\infty = 35^o\text{C}$

$T_f = (85 + 35)(^o\text{C})/2 = 60^o\text{C}$

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$k = 0.6507$ W/m-^{o}C

$Pr = 3.0$

$\nu = 0.4748 \times 10^{-6}$ m^2/s.

at $x = 7.5$ **cm** Re_x **and** Pe **are**

$$Re_x = \frac{V_\infty x}{\nu} = \frac{0.25(\text{m/s})0.075(\text{m})}{0.4748 \times 10^{-6}(\text{m}^2/\text{s})} = 3.949 \times 10^4$$

$$Pe = Re_x Pr = 3.949 \times 10^4 \times 3 = 11.85 \times 10^4$$

∴ **BL approximations are valid, flow is laminar Pohlhausen's solution is applicable.**

[a] Determine δ_t: **At** $y = \delta_t$, $T \approx T_\infty$

$$\theta(\eta_t) = \frac{(T_\infty - T_s)}{(T_\infty - T_s)} \approx 1$$

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From Fig. 4.6: Value of η_t **at** $\theta(\eta_t) = 1$ **and** $Pr = 3$ **at is approximately 2.9**

$$\eta_t \approx 2.9 = \delta_t \sqrt{V_\infty / \nu x}$$

or

$$\frac{\eta_t}{x} = \frac{2.9}{\sqrt{V_\infty / \nu x}} = \frac{2.9}{\sqrt{Re_x}} \qquad \text{(d)}$$

[b] Heat transfer coefficient:

$$h(x) = k\sqrt{\frac{V_\infty}{\nu x}}\frac{d\theta(0)}{d\eta} \qquad \text{(4.66)}$$

$\dfrac{d\theta(0)}{d\eta}$:

$$\frac{d\theta(0)}{d\eta} = 0.332\, Pr^{1/3}, \quad 0.6 < Pr < 10 \qquad \text{(4.71b)}$$

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Pr = 3

$$\frac{d\theta(0)}{d\eta} = 0.332(3)^{1/3} = 0.4788$$

Substituting into (4.66) for x = 0.075 m

$$h = 825.5 \frac{\text{W}}{\text{m}^2\text{–}^\circ\text{C}}$$

At x = 0.75 m

$$h = 261 \frac{\text{W}}{\text{m}^2\text{–}^\circ\text{C}}$$

[c] Heat transfer rate:

$$q_T = (T_s - T_\infty) A \bar{h} \qquad (4.70)$$

L = length of plate = 75 cm =0.75 m

W = width of plate = 50 cm = 0.5 m

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$$\bar{h} = 2\frac{k}{L}\sqrt{Re_L}\frac{d\theta(0)}{d\eta} \qquad (4.67)$$

$Re_L = 3.949 \times 10^5$. Substitute into the above

$$\bar{h} = 522.1 \frac{\text{W}}{\text{m}^2\text{–}^\circ\text{C}}$$

Substitute into (4.70)

$$q_T = 9789 \text{ W}$$

[d] Doubling the length of plate:

$$Re_{2L} = 2\,(3.949 \times 10^5) = 7.898 \times 10^5$$

$\therefore\ Re_{2L} > Re_t$

Flow is turbulent, Pohlhausen's solution is not applicable

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(iii) Checking. *Dimensional check*:

Reynolds number is dimensionless and that units of h and $\bar{h}$ are correct

Qualitative check: As x is increased h decreases

Quantitative check: Computed values of h are within the range of Table 1.1

(5) Comments

- Check Reynolds number before applying Pohlhausen's solution
- Velocity boundary layer thickness δ is given by

$$\frac{\delta}{x} = \frac{5.2}{\sqrt{Re_x}} \qquad (4.46)$$

Compare (d) with equation (4.46):

$$\delta_t < \delta$$

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Example 7.3: Scaling Estimate of Heat Transfer Rate

Use scaling to determine the total heat transfer rate for conditions described in Example 7.2

(1) Observation

- Newton's law gives heat transfer rate
- The heat transfer coefficient can be estimated using scaling

(2) Problem Definition.

Determine the heat transfer coefficient h

(3) Solution Plan.

Apply Newton's law of cooling and use scaling to determine h

(4) Plan Execution

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(i) Assumptions

- Newtonian fluid
- two-dimensional
- negligible changes in kinetic and potential energy
- constant properties
- boundary layer flow
- steady state
- no dissipation
- no gravity
- no energy generation
- no radiation

(ii) Analysis. Application of Newton's law of cooling gives

$$q_T = (T_s - T_\infty) A \bar{h} \tag{4.70}$$

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A = surface area = LW, m^2

$\bar{h}$ = average heat transfer coefficient, W/m^2-^{o}C

L = length of plate = 75 cm =0.75 m

q_T = total heat transfer rate from plate, W

T_s = surface temperature = 85^{o}C

T_∞ = free stream temperature = 35^{o}C

W = width of plate = 50 cm = 0.5 m

h by (1.10)

$$h = -k \frac{\dfrac{\partial T(x,0)}{\partial y}}{T_s - T_\infty} \tag{1.10}$$

k = thermal conductivity = 0.6507 W/m-^{o}C

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Follow analysis of Section 4.41, scale of h for Pr>>1

$$h \sim \frac{k}{x} Pr^{1/3} \sqrt{Re_x} \quad , \quad \text{for } Pr >> 1 \tag{4.56}$$

$$Re_x = \frac{V_\infty x}{\nu} \text{ and } Pr = 3$$

Set $h \sim \bar{h}$, $x = L$, $A = WL$ and substitute (4.56) into (4.70)

$$q_T \sim (T_s - T_\infty) W\, k\, Pr^{1/3} \sqrt{Re_L} \tag{a}$$

(iii) Computations

$$Re_L = 3.949 \times 10^5$$

Substitute into (a)

$$q_T \sim (85-35)(^{\circ}\text{C})0.5(\text{m})0.6507(\text{W/m}-^{\circ}\text{C})3^{1/3}\sqrt{394900}$$

$$q_T \sim 14740 \text{ W}$$

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Using Pohlhausen's solution gives $q_T = 9789$ W

(iv) Checking.

Dimensional Check: Solution (a) is dimensionally correct

(5) Comments.

Scaling gives an order of magnitude estimate of the heat transfer coefficient. In this example the error using scaling rate is 50%

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4.4.3 Laminar Boundary Layer Flow over Semi-infinite Flat Plate: Variable Surface Temperature

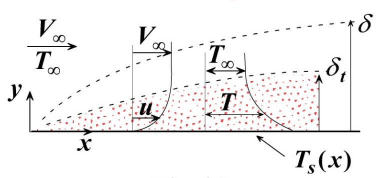

Fig. 4.7

- Consider uniform flow over plate
- Surface temperature varies with x as:

$$T_s(x) - T_\infty = Cx^n \tag{4.72}$$

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- C and n, constants
- T_∞ is free stream temperature
- Determine $T(x,y)$, $h(x)$, Nu_x and q_T
- Assumptions: summarized in Section 4.3

(i) Velocity Distribution

- For constant properties velocity is independent of the temperature distribution
- Blasius solution is applicable:

$$\frac{u}{V_\infty} = \frac{df}{d\eta} \tag{4.42}$$

$$\frac{v}{V_\infty} = \frac{1}{2}\sqrt{\frac{\nu}{V_\infty x}}\left(\eta \frac{df}{d\eta} - f\right) \tag{4.43}$$

$$\eta(x,y) = y\sqrt{\frac{V_\infty}{\nu\, x}} \tag{4.41}$$

(ii) Governing Equations for Temperature Distribution

Based on assumptions OF Section 4.3:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{4.18}$$

Boundary condition

$$T(x,0) = T_s = T_\infty + Cx^n \tag{4.73a}$$

$$T(x,\infty) = T_\infty \tag{4.73b}$$

$$T(0,y) = T_\infty \tag{4.73c}$$

(iii) Solution

- Solution to (4.18) is by similarity transformation

Define : θ

$$\theta = \frac{T - T_s}{T_\infty - T_s} \tag{4.58}$$

Assume

$$\theta(x,y) = \theta(\eta) \tag{4.75}$$

Use (4.41)-(4.43), (4.58), (4.72), (4.75), energy (4.18) transforms to (Appendix C)

$$\frac{d^2\theta}{d\eta^2} + nPr\frac{df}{d\eta}(1-\theta) + \frac{Pr}{2}f(\eta)\frac{d\theta}{d\eta} = 0 \tag{4.76}$$

B.C. (4.73):

$$\theta(\infty) = 1 \tag{4.76a}$$

$$\theta(0) = 0 \tag{4.76b}$$

$$\theta(\infty) = 1 \tag{4.76c}$$

• Note: Two B.C. coalesce into one

Heat transfer coefficient and Nusselt number:

Use (1.10)

$$h = -k\frac{\dfrac{\partial T(x,0)}{\partial y}}{T_s - T_\infty} \tag{1.10}$$

where

$$\frac{\partial T(x,0)}{\partial y} = \frac{dT}{d\theta}\frac{d\theta(0)}{d\eta}\frac{\partial \eta}{\partial y}$$

Use (4.41),(4.58) and (4.72) into the above

$$\frac{\partial T(x,0)}{\partial y} = -Cx^n\sqrt{\frac{V_\infty}{\nu x}}\frac{d\theta(0)}{d\eta}$$

Substitute into (1.10)

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$$h(x) = k\sqrt{\frac{V_\infty}{\nu x}}\frac{d\theta(0)}{d\eta} \tag{4.78}$$

• **Average heat transfer coefficient:** Use (2.50)

$$\bar{h} = \frac{1}{L}\int_0^L h(x)dx \tag{2.50}$$

Substitute (4.78) into (2.50) and integrate

$$\bar{h} = 2\frac{k}{L}\sqrt{Re_L}\frac{d\theta(0)}{d\eta} \tag{4.79}$$

• **Local Nusselt number:** (4.78) into (4.54)

$$Nu_x = \frac{d\theta(0)}{d\eta}\sqrt{Re_x} \tag{4.80}$$

• **Average Nusselt number:**

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$$\overline{Nu_L} = 2\frac{d\theta(0)}{d\eta}\sqrt{Re_L} \tag{4.81}$$

Heat transfer coefficient and Nusselt number depend on surface temperature gradient $\dfrac{d\theta(0)}{d\eta}$

(ii) Results:

• Equation (4.76) subject to boundary conditions (4.77) is solved numerically

• Solution depends on two parameters: the Prandtl number Pr and the exponent n in (4.72) $d\theta(0)/d\eta$ is presented in Fig. 4.8 for three Prandtl numbers.

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$\frac{d\theta(0)}{d\eta}$

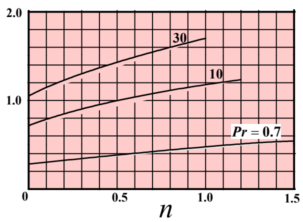

Fig. 4.8 $\frac{d\theta(0)}{d\eta}$ for plate with varying surface temperature $T_s(x) - T_\infty = C x^n$

4.4.3 Laminar Boundary Layer Flow over a Wedge: Uniform Surface Temperature

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- Symmetrical flow over a wedge of angle $\beta\pi$
- Uniform surface temperature
- Uniform upstream velocity, pressure and temperature

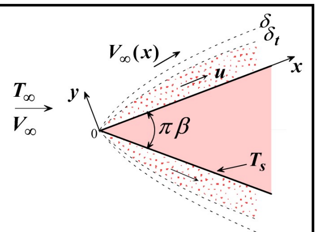

Fig. 4.9

- Both pressure and velocity outside the viscous BL vary with distance x along wedge
- For assumptions of Section 4.3, the x-momentum eq. is

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp_\infty}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \tag{4.13}$$

C is a constant and m describes wedge angle:

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$$m = \frac{\beta}{2-\beta} \tag{4.83}$$

- Apply (4.13) at edge of BL to determine $\frac{dp_\infty}{dx}$:
  - Flow is inviscid
  - $v = \nu = 0$
  - $u = V_\infty(x)$

$$-\frac{1}{\rho}\frac{dp_\infty}{dx} = V_\infty\frac{\partial V_\infty}{\partial x}$$

Substitute into (4.13)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = V_\infty\frac{\partial V_\infty}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2} \tag{4.84}$$

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The B.C. are

$$u(x,\infty)=V_\infty(x)=Cx^m \quad (4.84a)$$

$$u(x,0)=0 \quad (4.84b)$$

$$v(x,0)=0 \quad (4.84c)$$

(i) Velocity Solution:

- By similarity transformation (follow Blasius approach)
- Define a similarity variable η :

$$\eta(x,y)=y\sqrt{\frac{V_\infty(x)}{\nu x}}=y\sqrt{\frac{C}{\nu}}\,x^{(m-1)/2} \quad (4.86)$$

- Assume $u(x, y)$ to depend on η :

$$\frac{u}{V_\infty(x)}=\frac{dF}{d\eta} \quad (4.87)$$

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Continuity (2.2), (4.86) and (4.87) give v

$$v=-V_\infty(x)\sqrt{\frac{\nu}{xV_\infty(x)}}\frac{m+1}{2}\left[F-\frac{1-m}{1+m}\eta\frac{dF}{d\eta}\right] \quad (4.88)$$

Substitute (4.82) and (4.86)-(4.88) into (4.84)

$$\frac{d^3F}{d\eta^3}+\frac{m+1}{2}F\frac{d^2F}{d\eta^2}-m\left[\frac{dF}{d\eta}\right]^2+m=0 \quad (4.89)$$

This is the transformed momentum equation B. C. (4.85) transform to

$$\frac{dF(0)}{d\eta}=0 \quad (4.89a)$$

$$F(0)=0 \quad (4.89b)$$

$$\frac{dF(\infty)}{d\eta}=1 \quad (4.89c)$$

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Note the following regarding (4.89) and (4.90):

- x and y do not appear
- Momentum eq. (4.89) is 3rd order non-linear
- Special case: $m=\beta=0$ represents a flat plate
- Setting $m=0$ in (4.89) and (4.90) reduces to Blasius problem (4.44) & (4.45), $F(\eta)=f(\eta)$
- (4.89) is integrated numerically
- Solution gives $F(\eta)$ and $dF/d\eta$. These give u and v

(ii) Temperature Solution:

Energy equation:

$$u\frac{\partial\theta}{\partial x}+v\frac{\partial\theta}{\partial y}=\alpha\frac{\partial^2\theta}{\partial y^2} \quad (4.59)$$

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Boundary conditions:

$$\theta(x,0) = 0 \qquad (4.60a)$$

$$\theta(x,0) = 1 \qquad (4.60b)$$

$$\theta(x,0) = 1 \qquad (4.60c)$$

where

$$\theta = \frac{T - T_s}{T_\infty - T_s} \qquad (4.58)$$

- Same energy equation and B.C. as the flat plate.
- Is temperature distribution the same?
- Equation (4.59) is solved by similarity transformation. Assume:

$$\theta(x,y) = \theta(\eta) \qquad (4.75)$$

where

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$$\eta(x,y) = y\sqrt{\frac{V_\infty(x)}{\nu x}} = y\sqrt{\frac{C}{\nu}}\, x^{(m-1)/2} \qquad (4.86)$$

Substitute (4.86)-(4.88) and (4.75) into (4.59) and (4.60)

$$\frac{d^2\theta}{d\eta^2} + \frac{Pr}{2}(m+1)F(\eta)\frac{d\theta}{d\eta} = 0 \qquad (4.91)$$

$$\theta(0) = 0 \qquad (4.92a)$$

$$\theta(\infty) = 1 \qquad (4.92b)$$

$$\theta(\infty) = 1 \qquad (4.92c)$$

- Partial differential equations is transformed into ordinary equation
- Two governing parameters: Prandtl number Pr and the wedge size m
- (491) a linear second order equation requiring two B.C.

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- $F(\eta)$ in (4.91) represents effect of fluid motion
- B.C. (4.60b) and (4.60c) coalesce into a single condition
- Special case: $m = \beta = 0$ represents flat plate. Set $m = 0$ in (4.91) reduces to Pohlhausen's problem (4.61)

Solution: (Details in Appendix B)

- Separate variables in (4.91)
- Integrate twice
- Applying B.C. (4.92), gives

$$\theta(\eta) = 1 - \frac{\int_\eta^\infty \exp\left[-\frac{(\mathrm{m}+1)Pr}{2}\int_0^\eta F(\eta)d\eta\right] d\eta}{\int_0^\infty \exp\left[-\frac{(\mathrm{m}+1)Pr}{2}\int_0^\eta F(\eta)d\eta\right] d\eta} \qquad (4.93)$$

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Temperature gradient at surface $\dfrac{d\theta(0)}{d\eta}$:

- Differentiate (4.93), evaluate at $\eta = 0$

$$\frac{d\theta(0)}{d\eta} = \left\{ \int_0^\infty \exp\left[-\frac{(m+1)Pr}{2} \int_0^\eta F(\eta) d\eta \right] d\eta \right\}^{-1} \quad (4.94)$$

- $F(\eta)$ is given in the velocity solution
- Evaluate integrals in (4.93)&(4.94) numerically
- Results for $\dfrac{d\theta(0)}{d\eta}$ and $F''(0)$ are in Table 4.3

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Table 4.3 Surface temperature gradient $\dfrac{d\theta(0)}{d\eta}$ and velocity gradient $F''(0)$ for flow over an isothermal wedge

m	wedge angle $\pi\beta$	$F''(0)$	$d\theta(0)/d\eta$ at five values of Pr				
			0.7	0.8	1.0	5.0	10.0
0	0	0.3206	0.292	0.307	0.332	0.585	0.730
0.111	$\pi/5$ (36°)	0.5120	0.331	0.348	0.378	0.669	0.851
0.333	$\pi/2$ (90°)	0.7575	0.384	0.403	0.440	0.792	1.013
1.0	π (180°)	1.2326	0.496	0.523	0.570	1.043	1.344

- Use Table 4.3 to determine $h(x)$ and Nu_x

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$$h = -k \frac{\dfrac{\partial T(x,0)}{\partial y}}{T_s - T_\infty} \quad (1.10)$$

where

$$\frac{\partial T(x,0)}{\partial y} = \frac{dT}{d\theta} \frac{d\theta(0)}{d\eta} \frac{\partial \eta}{\partial y}$$

Use (4.58),(4.75) and (4.86) into above

$$\frac{\partial T(x,0)}{\partial y} = (T_\infty - T_s) \sqrt{\frac{V_\infty(x)}{\nu x}} \frac{d\theta(0)}{d\eta}$$

Substitute into (1.10)

$$h(x) = k \sqrt{\frac{V_\infty(x)}{\nu x}} \frac{d\theta(0)}{d\eta} \quad (4.95)$$

Local Nusselt number: substitute (4.95) into (4.54)

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$$Nu_x = \frac{d\theta(0)}{d\eta}\sqrt{Re_x} \qquad (4.96)$$

where

$$Re_x = \frac{xV_\infty(x)}{\nu} \qquad (4.97)$$

- Key factor in determining $h(x)$ and Nu_x :

  Surface temperature gradient is $\dfrac{d\theta(0)}{d\eta}$, listed in Table 4.3.

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