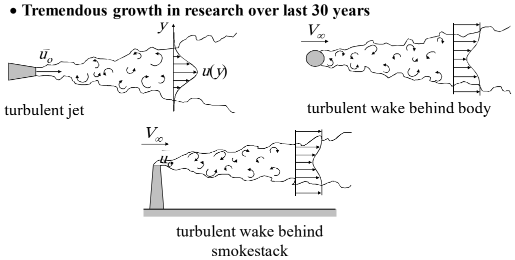
CHAPTER 8

CONVECTION IN EXTERNAL TURBULENT FLOW

8.1 Introduction

- **Common physical phenomenon, but complex**
- **Still relies on empirical data and rudimentary conceptual drawings**
- **Tremendous growth in research over last 30 years**



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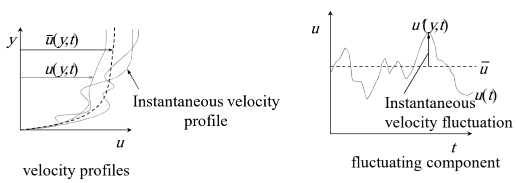
8.1.1 Examples of Turbulent Flows

(i) Mixing Processes

(ii) Free Shear Flows

(iii) Wall-Bounded Flows

- **Varying shape of instantaneous velocity profile**
- **Instantaneous velocity fluctuation**



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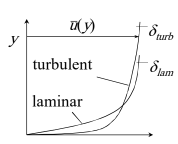
Turbulent velocity profile vs. laminar

- **Time averaged**



Turbulent flows can enhance performance

- **Turbulators**
- **Dimpled golf balls**

8.1.2 The Reynolds Number and the Onset of Turbulence

- Osborne Reynolds (1883) first identifies laminar and turbulent regimes
- Reynolds number:

$$Re_D = \frac{\bar{u}D}{\nu} \qquad (8.1)$$

- Internal flow: critical flow number is $Re_c = \bar{u}D/\nu \approx 2300$
- Flow over semi-infinite flat plate is $Re_c = V_\infty x_t / \nu \approx 500{,}000$

Why the Reynolds number predicts the onset of turbulence

- Reynolds number represents the ratio of inertial to viscous forces
  - o Inertial forces accelerate a fluid particle
  - o Viscous forces slow or damp the motion of the particle
- At low velocity, viscous forces dominate
  - o Infinitesimal disturbances damped out
  - o Flow remains laminar

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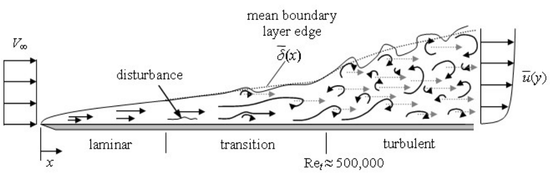
- At high enough fluid velocity, inertial forces dominate
  - o Viscous forces cannot prevent a wayward particle from motion
  - o Chaotic flow ensues

Turbulence near wall

- For wall-bounded flows, turbulence initiates near the wall



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8.1.3 Eddies and Vorticity

- An eddy is a particle of vorticity, ω,

$$\vec{\omega} = \nabla \times \vec{V} \qquad (8.2)$$

- Eddies typically form in regions of velocity gradient.
- Vorticity can be found from Eqn. (8.2) to be

$$\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

A Common View of Eddy Formation

- Eddy begins as a disturbance near the wall
- ***Vortex filament*** forms
- Stretched into ***horseshoe*** or ***hairpin*** vortex
- Lifting phenomenon

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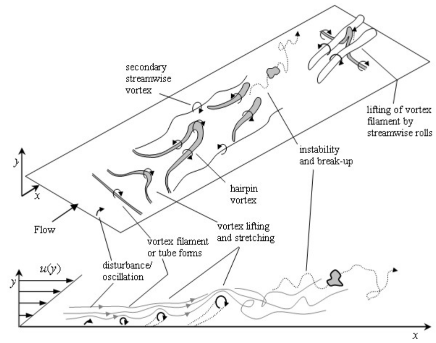

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8.1.4 Scales of Turbulence

- Largest eddies break up due to inertial forces
- Smallest eddies dissipate due to viscous forces
- Richardson Energy Cascade (1922)

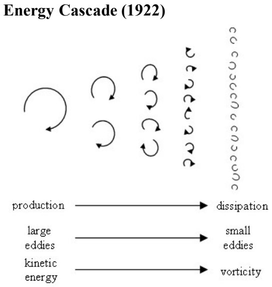

Fig. 8.7

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Kolgomorov Microscales (1942)

- **Attempt to estimate size of smallest eddies**

$$\eta / l \sim Re^{-3/4} \quad (8.3a)$$

$$v / u \sim Re^{-1/4} \quad (8.3b)$$

$$\tau / t \sim Re^{-1/2} \quad (8.3c)$$

- **Important impacts:**
  - o There is a vast range of eddy sizes, velocities, and time scales in a turbulent flow. This could make modeling difficult.
  - o The smallest eddies small, but not infinitesimally small. Viscosity dissipates them into heat before they can become too small.
  - o Scale of the smallest eddies are determined by the scale of the largest eddies through the Reynolds number. Generating smaller eddies is how the viscous dissipation is increased to compensate for the increased production of turbulence.

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8.1.5 Characteristics of Turbulence

- **Turbulence is comprised of irregular, chaotic, three-dimensional fluid motion, but containing coherent structures.**
- **Turbulence occurs at high Reynolds numbers, where instabilities give way to chaotic motion.**
- **Turbulence is comprised of many scales of eddies, which dissipate energy and momentum through a series of scale ranges. The largest eddies contain the bulk of the kinetic energy, and break up by inertial forces. The smallest eddies contain the bulk of the vorticity, and dissipate by viscosity into heat.**
- **Turbulent flows are not only dissipative, but also dispersive through the advection mechanism.**

8.1.6 Analytical Approaches

- **Considering small eddies, is continuum hypothesis still valid?**
  - **The smallest eddies: approximately 2×10^{-5} m**

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- **Mean free path of air at atmospheric pressure is on the order of 10^{-8} m three orders of magnitude smaller**
- **Continuum hypothesis OK**

- **Are numerical simulations possible?**
  - ***Direct Numerical Simulation* (DNS) a widespread topic of research**
  - **However, short time scales and size range of turbulence a problem**
  - **Still have to rely on more traditional analytical techniques**

Two Common Idealizations

- ***Homogeneous Turbulence:* Turbulence, whose microscale motion, on average, does not change from location to location and time to time.**
- ***Isotropic Turbulence:* Turbulence, whose microscale motion, on average, does not change as the coordinate axes are rotated.**

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8.2 Conservation Equations for Turbulent Flow

8.2.1 Reynolds Decomposition

- **Turbulent flow seems well-behaved *on average.***
- ***Reynolds Decomposition*: Separate velocity, properties into time-averaged and fluctuating components:**

$$g = \overline{g} + g' \qquad (8.4)$$

- **Time-averaged component is determined by:**

$$\overline{g} = \frac{1}{\tau}\int_0^{\tau} g(t)dt \qquad (8.5)$$

- **Time average of fluctuating component is zero:**

$$\overline{g'} = \frac{1}{\tau}\int_0^{\tau} g'(t)dt = 0 \qquad (8.6)$$

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Average Identities:

- For two variables $a = \bar{a} + a'$ and $b = \bar{b} + b'$

$$\bar{\bar{a}} = \bar{a} \quad (8.7a) \qquad \overline{\bar{a}b} = \bar{a}\bar{b} \quad (8.7b)$$

$$\overline{(\bar{a})^2} = (\bar{a})^2 \quad (8.7c) \qquad \overline{\bar{a}a'} = 0 \quad (8.7d)$$

$$\overline{ab} = \bar{a}\bar{b} + \overline{a'b'} \quad (8.7e) \qquad \overline{a^2} = (\bar{a})^2 + \overline{(a')^2} \quad (8.7f)$$

$$\overline{a+b} = \bar{a} + \bar{b} \quad (8.7g) \qquad \overline{\frac{\partial a}{\partial x}} = \frac{\partial \bar{a}}{\partial x} \quad (8.7h)$$

$$\frac{\partial \bar{a}}{\partial t} = 0 \quad (8.7i) \qquad \overline{\frac{\partial a}{\partial t}} = 0 \quad (8.7j)$$

8.2.2 Conservation of Mass

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- By identities (8.7a) and (8.6):

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \quad (2.2a)$$

- Assume incompressible, two-dimensional flow. Substituting the Reynolds-decomposed velocities $u = \bar{u} + u'$ and $v = \bar{v} + v'$,

$$\frac{\partial(\bar{u} + u')}{\partial x} + \frac{\partial(\bar{v} + v')}{\partial y} = 0 \quad (a)$$

- Expanding,

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial v'}{\partial y} = 0 \quad (b)$$

- Time-average the equation:

$$\overline{\frac{\partial \bar{u}}{\partial x}} + \overline{\frac{\partial u'}{\partial x}} + \overline{\frac{\partial \bar{v}}{\partial y}} + \overline{\frac{\partial v'}{\partial y}} = 0 \quad (c)$$

- Then, simplify each term by invoking identity (8.7h):

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$$\frac{\partial \bar{\bar{u}}}{\partial x} + \frac{\partial \overline{u'}}{\partial x} + \frac{\partial \bar{\bar{v}}}{\partial y} + \frac{\partial \overline{v'}}{\partial y} = 0 \quad (d)$$

- By identities (8.7a) and (8.6),

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \quad (8.8)$$

8.2.3 Conservation of Momentums

- The x and y momentum equations are given by:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) \quad (2.10x)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) \quad (2.10y)$$

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- **Simplifying for steady, 2D flow, no body forces:**

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad \textbf{(8.10x)}$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right) \qquad \textbf{(8.10y)}$$

- **For the *x*-momentum equation, the terms $u(\partial u/\partial x)$ and $v(\partial u/\partial y)$ can be replaced by the following relations, derived from the product rule of derivation:**

$$u\frac{\partial u}{\partial x}=\frac{\partial u^2}{\partial x}-u\frac{\partial u}{\partial x} \qquad \textbf{(a)}$$

$$v\frac{\partial u}{\partial y}=\frac{\partial (uv)}{\partial y}-u\frac{\partial v}{\partial y} \qquad \textbf{(b)}$$

- **Substitute (a) into the *x*-momentum equation (8.10x):**

$$\rho\left(\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}-\underbrace{u\frac{\partial u}{\partial x}}_{a}+\frac{\partial (uv)}{\partial y}-\underbrace{u\frac{\partial v}{\partial y}}_{b}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad \textbf{(c)}$$

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- **Note that terms marked ⓐ and ⓑ in the above can be combined as:**

$$-u\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0 \;\; \text{by continuity} \qquad \textbf{(d)}$$

- **Thus, the *x*-momentum equation reduces to:**

$$\rho\left(\frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial (uv)}{\partial y}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) \qquad \textbf{(8.11)}$$

- **Following Reynolds decomposition and averaging,**

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}\right)=-\frac{\partial \bar{p}}{\partial x}+\mu\left(\frac{\partial^2 \bar{u}}{\partial x^2}+\frac{\partial^2 \bar{u}}{\partial y^2}\right)-\rho\frac{\partial \overline{(u')^2}}{\partial x}-\rho\frac{\partial \overline{u'v'}}{\partial y} \qquad \textbf{(8.12x)}$$

$$\rho\left(\bar{u}\frac{\partial \bar{v}}{\partial x}+\bar{v}\frac{\partial \bar{v}}{\partial y}\right)=-\frac{\partial \bar{p}}{\partial y}+\mu\left(\frac{\partial^2 \bar{v}}{\partial x^2}+\frac{\partial^2 \bar{v}}{\partial y^2}\right)-\rho\frac{\partial \overline{u'v'}}{\partial x}-\rho\frac{\partial \overline{(v')^2}}{\partial y} \qquad \textbf{(8.12y)}$$

8.2.4 Conservation of Energy

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- **For incompressible flow, negligible heat generation, constant properties, the energy equation is given by**

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\mu\Phi \qquad \textbf{(2.19b)}$$

- **The energy equation reduces to:**

$$\rho c_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}\right) \qquad \textbf{(8.13)}$$

- **Following Reynolds decomposition and time averaging, Eqn. (8.13) becomes:**

$$\rho c_p\left(\bar{u}\frac{\partial \bar{T}}{\partial x}+\bar{v}\frac{\partial \bar{T}}{\partial y}\right)=k\left(\frac{\partial^2 \bar{T}}{\partial x^2}+\frac{\partial^2 \bar{T}}{\partial y^2}\right)-\rho c_p\frac{\partial\left(\overline{u'T'}\right)}{\partial x}-\rho c_p\frac{\partial\left(\overline{v'T'}\right)}{\partial y} \qquad \textbf{(8.14)}$$

- **NOTE TWO NEW TERMS**

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8.2.5 Summary of Governing Equations for Turbulent Flow

- **Continuity:**

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \qquad (8.8)$$

- ***x*-momentum:**

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = -\frac{\partial \bar{p}}{\partial x} + \mu\left(\frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2}\right) - \rho\frac{\partial \overline{(u')^2}}{\partial x} - \rho\frac{\partial \overline{u'v'}}{\partial y} \qquad (8.12x)$$

- ***y*-momentum:**

$$\rho\left(\bar{u}\frac{\partial \bar{v}}{\partial x} + \bar{v}\frac{\partial \bar{v}}{\partial y}\right) = -\frac{\partial \bar{p}}{\partial y} + \mu\left(\frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2}\right) - \rho\frac{\partial \overline{u'v'}}{\partial x} - \rho\frac{\partial \overline{(v')^2}}{\partial y} \qquad (8.12y)$$

- **Energy:**

$$\rho c_p\left(\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y}\right) = k\left(\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial y^2}\right) - \rho c_p\frac{\partial\left(\overline{u'T'}\right)}{\partial x} - \rho c_p\frac{\partial\left(\overline{v'T'}\right)}{\partial y}$$

(8.14)

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8.3 Analysis of External Turbulent Flow

8.3.1 Turbulent Boundary Layer Equations

(i) Turbulent Momentum Boundary Layer Equation

- **Consider a flat plate in turbulent flow.**
- **Assume boundary layer is thin:**

$$\frac{\delta}{L} \ll 1 \qquad (8.15)$$

- **Following the same arguments as for the laminar boundary layer, the following scalar arguments are made:**

$$\bar{u} \sim V_\infty \qquad (8.16a)$$

$$x \sim L \qquad (8.16b)$$

$$y \sim \delta \qquad (8.16c)$$

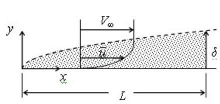

Fig. 8.8

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- **It can be shown that the viscous dissipation terms in (8.12x) compare as follows:**

$$\frac{\partial^2 \bar{u}}{\partial x^2} \ll \frac{\partial^2 \bar{u}}{\partial y^2} \qquad (8.17)$$

- **Also, the pressure gradient in the *y*-direction is negligible:**

$$\frac{\partial \bar{p}}{\partial y} \approx 0 \qquad (8.18)$$

- **The pressure gradient in the *x*-direction can be expressed as:**

$$\frac{\partial \bar{p}}{\partial x} = \frac{d\bar{p}}{dx} = \frac{dp_\infty}{dx} \qquad (8.19)$$

Simplifying the Fluctuation Terms:

- **Fluctuation Terms:**

$$\frac{\partial \overline{(u')^2}}{\partial x} \quad \text{and} \quad \frac{\partial \overline{u'v'}}{\partial y}$$

- **If fluctuation terms are the result of eddies, one could argue that there is no preferred direction to the fluctuations:**

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$$u' \sim v' \quad (8.20)$$

or

$$\overline{(u')^2} \sim \overline{u'v'} \quad (8.21)$$

- **Using scale analysis:**

First Fluctuation Term: $$\frac{\partial \overline{(u')^2}}{\partial x} \sim \frac{\overline{(u')^2}}{L} \quad (a)$$

Second Fluctuation Term: $$\frac{\partial \overline{u'v'}}{\partial y} \sim \frac{\overline{u'v'}}{\delta} \sim \frac{\overline{(u')^2}}{\delta} \quad (b)$$

- **Since $\delta / L \ll 1$, we conclude that:**

$$\frac{\partial \overline{(u')^2}}{\partial x} \ll \frac{\partial \overline{u'v'}}{\partial y} \quad (8.22)$$

- **The x-momentum equation for the turbulent boundary layer reduces to:**

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$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = -\frac{d\bar{p}}{dx} + \mu\frac{\partial^2 \bar{u}}{\partial y^2} - \rho\frac{\partial \overline{u'v'}}{\partial y} \quad (8.20)$$

(ii) Turbulent Energy Equation

- **Scaling arguments for the thermal boundary is:**

$$x \sim L \quad (8.16b)$$

$$y \sim \delta_t \quad (8.24)$$

$$\Delta T \sim T_s - T_\infty \quad (8.25)$$

- **Then:**

$$\frac{\partial^2 \bar{T}}{\partial x^2} \ll \frac{\partial^2 \bar{T}}{\partial y^2} \quad (8.26)$$

- **Fluctuation terms:**

$$\rho c_p \frac{\partial\left(\overline{u'T'}\right)}{\partial x} \text{ and } \rho c_p \frac{\partial\left(\overline{v'T'}\right)}{\partial y}$$

- **Assuming there is no preferred direction to the fluctuations:**

$$u' \sim v' \quad (8.20)$$

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or

$$\overline{u'T'} \sim \overline{v'T'} \quad (8.27)$$

- **We can then show that:**

$$\frac{\partial\left(\overline{u'T'}\right)}{\partial x} \ll \frac{\partial\left(\overline{v'T'}\right)}{\partial y} \quad (8.28)$$

- **The energy equation then reduces to:**

$$\rho c_p\left(\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y}\right) = k\frac{\partial^2 \bar{T}}{\partial y^2} - \rho c_p \frac{\partial\left(\overline{v'T'}\right)}{\partial y} \quad (8.29)$$

8.3.2 Reynolds Stress and Heat Flux

- **Can write the x-momentum and energy boundary layer equations as:**

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = -\frac{d\bar{p}}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right) \quad (8.30)$$

$$\rho c_p\left(\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial \bar{T}}{\partial y} - \rho c_p \overline{v'T'}\right) \quad (8.31)$$

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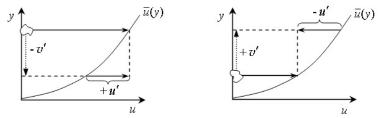
- Fluctuating term in (8.30) "looks" like a shear stress
- Fluctuating term in (8.31) "looks" like a heat flux
- Consider a particle "fluctuation" imposed on some average velocity profile



a. Particle motion toward wall   b. Particle motion away from wall

Fig. 8.9

- $\rho\overline{u'v'}$ is called the ***turbulent shear stress*** or the ***Reynolds stress***
- $\rho c_p\overline{v'T'}$ is called the ***turbulent heat flux*** or the ***Reynolds heat flux***

8.3.3 The Closure Problem of Turbulence

- Turbulent boundary layer equations:
- Continuity:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0 \qquad (8.8)$$

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- x-momentum:

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = -\frac{d\bar{p}}{dx} + \frac{\partial}{\partial y}\left(\mu\frac{\partial \bar{u}}{\partial y} - \rho\overline{u'v'}\right) \qquad (8.30)$$

- Energy:

$$\rho c_p\left(\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial \bar{T}}{\partial y} - \rho c_p\overline{v'T'}\right) \qquad (8.31)$$

- Boundary conditions:

$$\bar{u}(x,0) = 0 \qquad (8.31a)$$
$$\bar{v}(x,0) = 0 \qquad (8.31b)$$
$$\bar{u}(x,\infty) = V_\infty \qquad (8.31c)$$
$$\bar{u}(0,y) = V_\infty \qquad (8.31d)$$
$$\bar{T}(x,0) = T_s \qquad (8.31e)$$
$$\bar{T}(x,\infty) = T_\infty \qquad (8.31f)$$
$$\bar{T}(0,y) = T_\infty \qquad (8.31g)$$

- Also have, outside the boundary layer:

$$\frac{d\bar{p}}{dx} = \frac{dp_\infty}{dx} \qquad (8.32)$$

$$V_\infty\frac{dV_\infty}{dx} = -\frac{1}{\rho}\frac{dp_\infty}{dx} \qquad (8.33)$$

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- Leaves us with three equations (8.8), (8.30) and (8.31), but five unknowns:

$$\bar{u},\ \bar{v},\ \bar{T},\ \overline{u'v'} \text{ and } \overline{v'T'}$$

- This is the *closure problem of turbulence.*

8.3.4 Eddy Diffusivity

- Customary to model the Reynolds stress as

$$-\rho\overline{u'v'} = \rho\varepsilon_M\frac{\partial \bar{u}}{\partial y} \qquad (8.34)$$

- ε_M is called the ***momentum eddy diffusivity.*** $\rho\varepsilon_M$ is often referred to as ***eddy viscosity.***
- Similarly, we can model the Reynolds heat flux as

$$-\rho c_p\overline{v'T'} = \rho c_p\varepsilon_H\frac{\partial \bar{T}}{\partial y} \qquad (8.35)$$

- ε_H is called the ***thermal eddy diffusivity*** $\rho c_p\varepsilon_H$ is often referred to as ***eddy conductivity.***
- We can then write the boundary layer momentum and energy equations as

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$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}=\frac{\partial}{\partial y}\left[\left(\nu+\varepsilon_M\right)\frac{\partial \bar{u}}{\partial y}\right] \quad (8.38)$$

$$\bar{u}\frac{\partial \bar{T}}{\partial x}+\bar{v}\frac{\partial \bar{T}}{\partial y}=\frac{\partial}{\partial y}\left[\left(\alpha+\varepsilon_H\right)\frac{\partial \bar{T}}{\partial y}\right] \quad (8.39)$$

- **The terms in brackets represent the *apparent shear stress* and *apparent heat flux*, respectively:**

$$\frac{\tau_{app}}{\rho}=\left(\nu+\varepsilon_M\right)\frac{\partial \bar{u}}{\partial y} \quad (8.40)$$

$$-\frac{q''_{app}}{\rho c_p}=\left(\alpha+\varepsilon_H\right)\frac{\partial \bar{T}}{\partial y} \quad (8.41)$$

8.4 Momentum Transfer in External Turbulent Flow

8.4.1 Modeling Eddy Diffusivity: Prandtl's Mixing Length Theory

- **Simplest model by Boussinesq: constant ε_M**
  - o **does not allow $\overline{u'v'}$ to approach zero at the wall**

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- **Prandtl (1925): used analogy to kinetic theory of gases**
- **Define the mixing length l as the distance the particle travels towards the wall as the result of a fluctuation.**

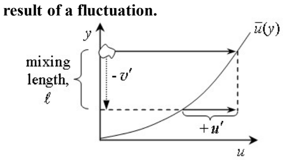

Fig. 8.10

- **The velocity fluctuation u' that results can be approximated from a Taylor series as**

$$u' \approx u_{initial}+\frac{\partial \bar{u}}{\partial y}dy \quad (a)$$

$$u' = u_{final}-u_{initial} \quad (a)$$

- **Thus,**

$$u' \sim \frac{\partial \bar{u}}{\partial y} l \quad (b)$$

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- **If we assume, as we have before, that fluctuations have no preferred direction, then $u' \sim v'$, and so**

$$v' \sim \frac{\partial \bar{u}}{\partial y} l \quad (c)$$

- **One could argue, then, that the turbulent stress term $-\overline{u'v'}$ is of the following scale:**

$$-\overline{u'v'} \sim \left(u'\right)\left(v'\right) \sim l^2\left(\frac{\partial \bar{u}}{\partial y}\right)^2 \quad (d)$$

- **Finally, we can solve Equation (8.34), for the eddy viscosity:**

$$\varepsilon_M=\frac{-\overline{u'v'}}{\partial \bar{u}/\partial y} \sim l^2\left|\frac{\partial \bar{u}}{\partial y}\right| \quad (8.42)$$

- **Prandtl proposed the following model for the mixing length,**

$$l=\kappa y \quad (8.43)$$

- **Leading to *Prandtl's mixing-length model*:**

$$\varepsilon_M=\kappa^2 y^2\left|\frac{\partial \bar{u}}{\partial y}\right| \quad (8.44)$$

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8.4.2 Universal Turbulent Velocity Profile

- One way to solve momentum is to assume a velocity profile, then use approximate methods to solve integral momentum (like in Chap. 5)

(i) Large-Scale Velocity Distribution: "Velocity Defect Law"

- First Step: normalize variables: $\bar{u}/V_\infty$ vs. y/δ
- Doesn't collapse curves with varying friction

Velocity Defect Law

- Introduce coefficient of friction:

$$C_f = \frac{\tau_o}{(1/2)\rho V_\infty^2} \qquad (8.45)$$

- Second, define a ***friction velocity*** as:

$$u^* \equiv \sqrt{\tau_o/\rho} \qquad (8.46)$$

$$u^* = V_\infty \sqrt{C_f/2} \qquad (8.47)$$

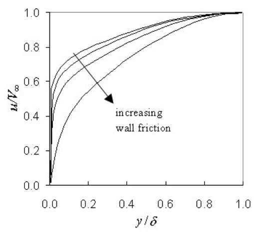


Fig. 8.11

- Define ***velocity defect***:

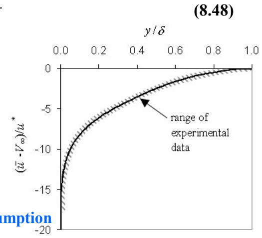
$$\frac{\bar{u} - V_\infty}{u^*} \qquad (8.48)$$

- This works, but doesn't provide enough detail near the wall.

(ii) Wall Coordinates

- Dimensional analysis suggests the following ***wall coordinates***:

$$u^+ \equiv \frac{\bar{u}}{u^*} \qquad (8.49)$$



(iii) Near-Wall Profile: Couette Flow Assumption

- Very close to the wall, scaling analysis suggests:

Near the Wall:
$$\frac{\partial}{\partial y}\left[(\nu + \varepsilon_M)\frac{\partial \bar{u}}{\partial y}\right] \approx 0 \qquad (8.50)$$

$$\frac{\tau_{app}}{\rho} = (\nu + \varepsilon_M)\frac{\partial \bar{u}}{\partial y} \approx \text{constant} \qquad (8.51)$$

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- This result is similar to Couette flow: ***Couette Flow Assumption***
- Resulting curve:

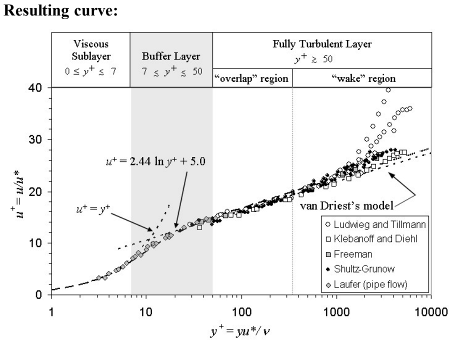

Fig. 8.13

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- What do we do with this? We can use (8.51) to develop an expression for the velocity profile.
- First, we need to express (8.51) in terms of the wall coordinates u^+ and y^+. Substituting their definitions, it can be shown that:

$$\left(1+\frac{\varepsilon_M}{\nu}\right)\frac{\partial u^+}{\partial y^+}=1 \qquad (8.52)$$

- And after rearranging and integrating,

$$u^+=\int_0^{y^+}\frac{dy^+}{\left(1+\varepsilon_M/\nu\right)} \qquad (8.53)$$

(iv) Viscous Sublayer

- Very close to the wall, viscous forces dominate, $\nu \gg \varepsilon_M$
- Couette Flow Assumption (8.52) reduces to:

$$\frac{\partial u^+}{\partial y^+}=1$$

- Integrating, with boundary condition $u^+=0$ at $y^+=0$

$$u^+=y^+,\quad (0\le y^+\le 7) \qquad (8.54)$$

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- This relation compares well to experimental data from $y^+\approx 0$ to 7 which we call the *viscous sublayer*.

(v) Fully Turbulent Region: "Law of the Wall"

- Further away from the wall, turbulent fluctuations dominate, $\varepsilon_M \gg \nu$
- Couette Flow Assumption (8.52) becomes

$$\frac{\varepsilon_M}{\nu}\frac{\partial u^+}{\partial y^+}=1 \qquad (8.55)$$

- Substitute Prandtl's mixing length, and wall coordinates:

$$\varepsilon_M=\kappa^2\left(y^+\right)^2\nu\frac{\partial u^+}{\partial y^+} \qquad (8.56)$$

- Substitute into Equation (8.55),

$$\kappa^2\left(y^+\right)^2\left(\frac{\partial u^+}{\partial y^+}\right)^2=1$$

- Solve for the velocity gradient,

$$\frac{\partial u^+}{\partial y^+}=\frac{1}{\kappa y^+} \qquad (8.57)$$

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- Finally, integrate the above to obtain

$$u^+=\frac{1}{\kappa}\ln y^+ + B \qquad (8.58)$$

- This is sometimes referred to as the *Law of the Wall.*
- The constant κ is called *von Karman's constant*, and experimental measurements show that $\kappa\approx 0.41$.
- The constant of integration B can be estimated by noting that the viscous sublayer and the Law of the Wall region appear to intersect at roughly $y^+=u^+\approx 10.8$. Using this as a boundary condition, the integration constant is found to be $\beta\approx 5.0$.
- Thus, an approximation for the Law of the Wall region is:

$$u^+=2.44\ln y^+ + 5.0 \qquad (50<y^+<1500) \qquad (8.59)$$

(vi) Other Models

- van Driest's *continuous law of the wall*
  - o van Driest proposed a mixing length model of this form:

$$l=\kappa y\left(1-e^{-y/A}\right) \qquad (8.60)$$

  - o van Driest used this equation with (8.42) and (8.51) to obtain:

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$$\frac{\tau_{app}}{\rho} = \left[\nu + \kappa^2 y^2 \left(1-e^{-y/A}\right)^2\right]\frac{\partial \bar{u}}{\partial y} \qquad (8.61)$$

- Transforming (8.61) into wall coordinates, and solving for $\partial y^+ / \partial u^+$ one can obtain:

$$\frac{\partial u^+}{\partial y^+} = \frac{2}{1+\sqrt{1+\kappa^2 y^{+2}\left(1-e^{-y^+/A^+}\right)^2}} \qquad (8.62)$$

- Spalding's Law:

$$y^+ = u^+ + e^{-\kappa\beta}\left[e^{\kappa u^+} - 1 - \kappa u^+ - \frac{\left(\kappa u^+\right)^2}{2} - \frac{\left(\kappa u^+\right)^3}{6}\right] \qquad (8.63)$$

  - o Works for flat plate and pipe flow
- Reichardt's Law, applied frequently to pipe flow:

$$u^+ = \frac{1}{\kappa}\ln\left(1+\kappa y^+\right) + C\left[1 - e^{-y^+/X} - \frac{y^+}{X}e^{-0.33 y^+}\right] \qquad (8.64)$$

(vii) Effect of Pressure Gradient

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- In the presence of an adverse pressure gradient, the velocity profile beyond $y^+ \approx 350$ deviates from the Law of the Wall model.
- The deviation is referred to as a "wake," and the region $y^+ > 350$ is commonly referred to as the *wake region*, where the velocity profile deviates even from the overlap region.
- Law of the Wall-type models developed earlier model flat plate flow reasonably well in the presence of zero pressure gradient.
- A favorable pressure gradient is approximately what we encounter in pipe flow, which helps explain why the models developed here apply as well to pipe flow.

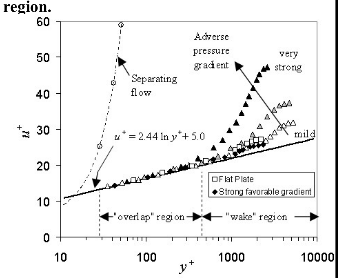

Fig. 8.14

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8.4.3 Approximate Solution for Momentum Transfer: Momentum Integral Method

(i) Prandtl - von Karman Model

- Consider a flat, impermeable plate exposed to incompressible, zero-pressure-gradient flow
- The integral momentum equation reduces to equation (5.5),

$$\nu\frac{\partial \bar{u}(x,0)}{\partial y} = V_\infty \frac{d}{dx}\int_0^{\delta(x)} \bar{u}\,dy - \frac{d}{dx}\int_0^{\delta(x)} \bar{u}^2 dy \qquad (5.5)$$

- Applies to turbulent flows as well – without modification – if we look at the behavior of the flow on average, and we interpret the flow properties as time-averaged values.

Estimate of Velocity Profile

- The integral method requires an estimate for the velocity profile in the boundary layer.
- Prandtl and von Kármán both used a crude but simple model for the velocity profile using prior knowledge about pipe flow.

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- Using the Blasius model for the shear at the wall of a circular pipe, Prandtl [18] and von Kármán [16] each showed that the velocity profile in the pipe could be modeled as

$$\frac{\bar{u}}{u_{CL}} = \left(\frac{y}{r_o}\right)^{1/7}$$

- This is the well-known *1/7th Law velocity profile,* discussed further in Chapter 9.
- Why base a velocity profile for flat plate on pipe flow? The velocity data for pipe flow and flat plate flow (at zero or favorable pressure gradient) have essentially the same shape, so the use of this model to describe flow over a flat plate is not unreasonable.
- To apply the 1/7th law to flat plate, we approximate r_o as the edge of the boundary layer δ , and approximate u_{CL} as V_∞ . Then,

$$\frac{\bar{u}}{V_\infty} = \left(\frac{y}{\delta}\right)^{1/7} \qquad (8.65)$$

Model for Wall Shear

- LHS of Integral Momentum is an expression for wall shear; uses assumed velocity profile.

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- Problem: our assumed profile goes to infinity as *y* approaches zero.
- To avoid this dilemma, Prandtl and von Kármán again looked to pipe flow knowledge
- They adapted the Blasius correlation for pipe flow friction factor to find an expression for the wall shear on a flat plate
- Recasting the Blasius correlation terms of the wall shear and the tube radius, they obtained

$$\frac{C_f}{2} = \frac{\tau_o}{\rho V_\infty^2} = 0.02333\left(\frac{V_\infty \delta}{\nu}\right)^{-1/4} \qquad (8.67)$$

- This is used in the LHS of the momentum integral relation.

Example 8.2: Integral Solution for Turbulent Boundary Layer Flow over a Flat Plate

Consider turbulent flow over a flat plate, depicted in Fig. 8.8. Using the 1/7th law velocity profile (8.65) and the expression for friction factor (8.67), obtain expressions for the boundary layer thickness and friction factor along the plate.

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(1) Observations. The solution parallels that of Chapter 5 for laminar flow over a flat plate.

(2) Problem Definition. Determine expressions for the boundary layer thickness and friction factor as a function of *x*.

(3) Solution Plan. Start with the integral Energy Equation (5.5), substitute the power law velocity profile (8.65) and friction factor (8.67), and solve.

(4) Plan Execution.

(i) Assumptions. (1) Boundary layer simplifications hold, (2) constant properties, (3) incompressible flow, (4) impermeable flat plate.

(ii) Analysis.

- Substitute 1/7th power law velocity profile into the Mom. Int. Equation:

$$\frac{\tau_o}{\rho} = V_\infty \frac{d}{dx}\int_0^{\delta(x)} V_\infty \left(\frac{y}{\delta}\right)^{1/7} dy - \frac{d}{dx}\int_0^{\delta(x)} V_\infty^2 \left(\frac{y}{\delta}\right)^{2/7} dy \qquad (a)$$

- Dividing the expression by V_∞^2 , and collecting terms,

$$\frac{\tau_o}{\rho V_\infty^2} = \frac{d}{dx}\int_0^{\delta(x)} \left[\left(\frac{y}{\delta}\right)^{1/7} - \left(\frac{y}{\delta}\right)^{2/7}\right] dy \qquad (b)$$

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- After integrating,

$$\frac{\tau_o}{\rho V_\infty^{\,2}} = \frac{7}{72}\frac{d\delta}{dx} \qquad (8.68)$$

- Now substituting (a) into the wall shear expression (8.67),

$$0.02333\left(\frac{V_\infty \delta}{\nu}\right)^{-1/4} = \frac{7}{72}\frac{d\delta}{dx} \qquad (c)$$

- Then, separating variables and integrating,

$$\frac{4}{45}\delta^{5/4} = 0.02333\left(\frac{72}{7}\right)\left(\frac{V_\infty}{\nu}\right)^{-1/4} x + C \qquad (8.69)$$

- To complete the solution, a boundary condition is needed.
- Can assume that $\delta(x)$ is zero at $x = 0$, which ignores the initial laminar boundary layer region
- However crude the assumption, we find that the results of this analysis compare well to experimental data.

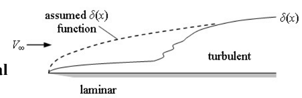

Fig. 8.15

- With the boundary condition established, the integration constant C equals zero. Then solving (8.69) for $\delta(x)$,

$$\delta(x) = 0.3816\left(\frac{V_\infty x}{\nu}\right)^{-1/5} x \qquad (d)$$

or

$$\frac{\delta}{x} = \frac{0.3816}{Re_x^{\,1/5}} \qquad (8.70)$$

- Finally, solve for the friction factor. Substituting (8.70) into (8.67),

$$\frac{C_f}{2} = 0.02333\left(\frac{0.3816 V_\infty \left(\frac{V_\infty x}{\nu}\right)^{-1/5} x}{\nu}\right)^{-1/4} \qquad (e)$$

Which reduces to

$$\frac{C_f}{2} = \frac{0.02968}{Re_x^{\,1/5}} \qquad (8.71)$$

(5) Checking. Equations (8.70) and (8.71) are both dimensionless, as expected.

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(6) Comments. Note that, according to this model, the turbulent boundary layer δ/x varies as $Re_x^{-1/5}$, as does the friction factor C_f. This is contrast to laminar flow, in which δ/x and C_f vary as $Re_x^{-1/2}$.

(ii) Newer Models

- One limitation of the Prandtl-von Kármán model is that the approximation for the wall shear, Eqn. (8.66), is based on limited experimental data, and considered to be of limited applicability even for pipe flow.

White's Model

- White [14] uses the Law of the Wall velocity profile (8.59) to model the wall shear.
- First, substituting the definitions of u^+ and y^+, as well as u^*, into the Law of the Wall expression (8.59),

$$\frac{\bar{u}}{V_\infty}\sqrt{\frac{2}{C_f}} = 2.44\ln\left(\frac{yV_\infty}{\nu}\sqrt{\frac{C_f}{2}}\right)^{-1/4} + 5.0$$

- In theory, any y value within the wall law layer would satisfy this expression, but a useful value to choose is the edge of the boundary layer, where $\bar{u}(y=\delta) = V_\infty$. Then, the above can be expressed as

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$$\frac{1}{\sqrt{C_f/2}} = 2.44\ln\left(Re_\delta\sqrt{\frac{C_f}{2}}\right) + 5.0 \quad (8.72)$$

- Still a difficult relation to use, but a simpler curve fit over a range of values from $Re_\delta \approx 10^4$ to 10^7 gives

$$C_f \approx 0.02 Re_\delta^{-1/6} \quad (8.73)$$

- We can now use this expression to estimate the wall shear in the integral method.
- For the velocity profile, the 1/7th power law is still used.
- It can be shown that the solution to the momentum integral equation in this case becomes

$$\frac{\delta}{x} = \frac{0.16}{Re_\delta^{1/7}} \quad (8.74)$$

and

$$\frac{C_f}{2} = \frac{0.0135}{Re_\delta^{1/7}} \quad (8.75)$$

- Equations (8.74) and (8.75) replace the less accurate Prandtl-von Kármán correlations, and White recommends these expressions for general use.

Kestin and Persen's Model

- Perhaps a more accurate correlation would result if we use one of the more advanced velocity profiles to estimate the wall shear, as well as to replace the crude 1/7th power law profile.
- Kestin and Persen used Spalding's law of the wall for the velocity profile and shear stress.
- The resulting model is extremely accurate, but cumbersome. White [20] modified the result to obtain the simpler relation

$$C_f = \frac{0.455}{\ln^2\left(0.06 Re_x\right)} \quad (8.76)$$

- White reports that this expression is accurate to within 1% of Kestin and Persen's model.

(iii) Total Drag

- The total drag is found by integrating the wall shear along the entire plate. Assuming the presence of an initial laminar flow region,

$$F_D = \int_0^{x_{crit}} (\tau_o)_{lam}\, w dx + \int_{x_{crit}}^{L} (\tau_o)_{turb}\, w dx \quad (8.76)$$

- Dividing by $\frac{1}{2}\rho V_\infty^2 A = \frac{1}{2}\rho V_\infty^2 wL$, the drag coefficient C_D is:

$$C_D = \frac{1}{L}\left[\int_0^{x_{crit}} C_{f,lam}\, dx + \int_{x_{crit}}^{L} C_{f,turb}\, dx\right] \quad (8.78)$$

- Substituting Eqn. (4.48) for laminar flow and using White's model (8.75) for turbulent flow, we obtain with some manipulation,

$$C_D = \frac{0.0315}{Re_L^{1/7}} - \frac{1477}{Re_L} \quad (8.79)$$

- Assume $x_{crit} = 5\times10^5$

8.4.4 Effect of Surface Roughness on Friction Factor

- The interaction between the already complex turbulent flow and the complex, random geometric features of a rough wall is the subject of advanced study and numerical modeling.
- However, with crude modeling and some experimental study we can gain at least some physical insight.
- Define k as the average height of roughness elements on the surface. In wall coordinates:

$$k^+ = ku^*/\nu$$

- Experiments show that for small values of k^+ (less than approximately 5), the velocity profile and friction factor are unaffected by roughness
- For $k^+ > 10$ or so, however, the roughness extends beyond the viscous sublayer, and the viscous sublayer begins to disappear, likely due to the enhanced mixing in the roughness provided.
- Beyond $k^+ > 70$ viscous effects are virtually eliminated, and the flow is referred to as ***fully rough.*** Beyond this value of roughness, the shape of the velocity profile changes very little. Consequently, we might expect that once the surface is fully rough, increasing the roughness would not change the friction factor.

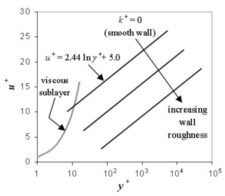

Fig. 8.16

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8.5 Energy Transfer in External Turbulent Flow

- Not surprisingly, energy transfer is also greatly complicated under turbulent flow.
- We found in Chapter 2 that the heat transfer for flow over a geometrically similar body like a flat plate (neglecting both buoyancy and viscous dissipation) could be correlated through dimensionless analysis by

$$Nu_x = f(x^*, Re, Pr) \qquad (2.52)$$

- Turbulence introduces two new variables into the analysis: the momentum and thermal eddy diffusivities, ε_M and ε_H.
- One way to deal with these new terms is to introduce a new dimensionless parameter: ***Turbulent Prandtl Number***

$$Pr_t = \frac{\varepsilon_M}{\varepsilon_H} \qquad (8.81)$$

Approaches to Analyzing Turbulent Heat Transfer

- Find a mathematical analogy between heat and mass transfer
- Develop a universal temperature profile, similar to how we developed a universal velocity profile.

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  - o Then attempt to obtain an approximate solution for heat transfer using the integral method
- The universal temperature profile may also lend itself to a simple algebraic method for evaluating the heat transfer.
- There are more advanced methods, like numerical solutions to the boundary layer flow, which we will forgo in this text. We will instead remain focused on some of the more traditional methods, which are the basis of the correlations commonly in use.

8.5.1 Momentum and Heat Transfer Analogies

- Osborne Reynolds first discovered a link between momentum and heat transfer in 1874 while studying boilers.
- He theorized that the heat transfer and the frictional resistance in a pipe are proportional to each other.
- This is a significant and bold assertion! If we can measure or predict the friction along a wall or pipe, we can determine the heat transfer simply by using a multiplying factor. This approach would allow us to solve for the heat transfer directly, avoiding the difficulty of solving the energy equation.

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(i) Reynolds Analogy

- **Consider parallel flow over a flat plate.**
- **The pressure gradient *dp/dx* is zero, and the boundary layer momentum and energy equations (8.38) and (8.39) reduce to**

$$\bar{u}\frac{\partial \bar{u}}{\partial x}+\bar{v}\frac{\partial \bar{u}}{\partial y}=\frac{\partial}{\partial y}\left[\left(\nu+\varepsilon_M\right)\frac{\partial \bar{u}}{\partial y}\right] \quad (8.82a)$$

$$\bar{u}\frac{\partial \bar{T}}{\partial x}+\bar{v}\frac{\partial \bar{T}}{\partial y}=\frac{\partial}{\partial y}\left[\left(\alpha+\varepsilon_H\right)\frac{\partial \bar{T}}{\partial y}\right] \quad (8.82b)$$

- **The boundary conditions are**

$$\bar{u}(y=0)=0,\ \bar{T}(y=0)=T_s \quad (8.83a)$$

$$\bar{u}(y\to\infty)=V_\infty,\ \bar{T}(y\to\infty)=T_\infty \quad (8.83b)$$

- **Notice that equations (8.82a&b) and their respective boundary conditions are very similar; if they were identical, their solutions – the velocity and temperature profiles – would be the same.**

Normalizing the Variables

- **Select the following variables,**

$$U=\frac{\bar{u}}{V_\infty},\ V=\frac{\bar{v}}{V_\infty},\ \theta=\frac{\bar{T}-T_s}{T_\infty-T_s},\ X=\frac{x}{L}\ \text{and}\ Y=\frac{y}{L}$$

- **The boundary layer equations become**

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=\frac{1}{V_\infty L}\frac{\partial}{\partial Y}\left[\left(\nu+\varepsilon_M\right)\frac{\partial U}{\partial Y}\right] \quad (8.84a)$$

$$U\frac{\partial \theta}{\partial X}+V\frac{\partial \theta}{\partial Y}=\frac{1}{V_\infty L}\frac{\partial}{\partial y}\left[\left(\alpha+\varepsilon_H\right)\frac{\partial \theta}{\partial Y}\right] \quad (8.84b)$$

- **With boundary conditions:**

$$U(Y=0)=0,\ \theta(Y=0)=0 \quad (8.85a)$$

$$U(Y\to\infty)=1,\ \theta(Y\to\infty)=1 \quad (8.85b)$$

- **Normalizing the variables has made the boundary conditions identical.**
- **The boundary layer equations (8.84) can then be made identical if $\nu+\varepsilon_M=\alpha+\varepsilon_H$ which is possible under two conditions.**
  1. **The kinematic viscosity and thermal diffusivity are equal:**

$$\nu=\alpha\ \ (Pr=1) \quad (8.86)$$

  2. **The eddy diffusivities are equal:**

$$\varepsilon_M=\varepsilon_H\ \ (Pr_t=1) \quad (8.87)$$

- **We can provide some justification for this assumption by arguing that the same turbulent mechanism—the motion and interaction of fluid particles—is responsible for both momentum and heat transfer. Reynolds made essentially the same argument, and so Equation (8.87) by itself is sometimes referred to as Reynolds' analogy.**
- **The analogy is now complete, meaning that the normalized velocity and temperature profiles, $U(X,Y)$ and $V(X,Y)$ are equal.**

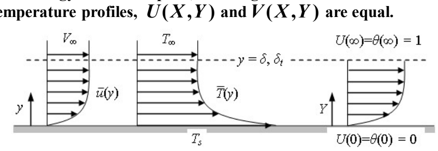

Fig. 8.17

Developing the Analogy

- **Begin by writing the ratio of the apparent heat flux and shear stress (equations. 8.40 and 8.41),**

$$\frac{q''_{app}/\rho c_p}{\tau_{app}/\rho} = -\frac{(\alpha+\varepsilon_H)\partial \bar{T}/\partial y}{(\nu+\varepsilon_M)\partial \bar{u}/\partial y} \qquad (8.88)$$

- **Imposing the two conditions $\nu = \alpha$ (8.86) and $\varepsilon_M = \varepsilon_H$ (8.87), substituting the dimensionless variables yields**

$$\frac{q''_{app}}{\tau_{app}} = \frac{c_p(T_s - T_\infty)\partial\theta/\partial Y}{V_\infty \partial U/\partial Y} \qquad (8.89)$$

- **Since the dimensionless velocity and temperature profiles are identical, their derivatives cancel.**
- **Another important implication of (8.89) is that the ratio q''_{app}/τ_{app} is constant throughout the boundary layer. This means we can represent this ratio by the same ratio at the wall. Equation (8.89) then becomes**

$$\frac{q''_o}{\tau_o} = \frac{c_p(T_s - T_\infty)}{V_\infty}$$

- **can recast this into a more convenient form by substituting $q_o'' = h(T_s - T_\infty)$ and $\tau_o = 0.5C_f\rho V_\infty^{\,2}$ into the above, and rearranging,**

$$\frac{h}{\rho V_\infty c_p} = \frac{C_f}{2}$$

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- **The terms on the left side can also be written in terms of the Reynolds, Nusselt and Prandtl numbers,**

$$St_x = \frac{Nu_x}{Re_x Pr} = \frac{C_f}{2}\ (Pr = 1) \qquad (8.90)$$

- **This is the *Reynolds Analogy.***
- ***St* is called the *Stanton Number.***
- ***Note:* The same analogy can also be derived for *laminar* flow over a flat plate (for *Pr*=1).**

Limitations to the Reynolds Analogy

- **It is limited to *Pr* =1 fluids.**
  - **A reasonable approximation for many gases, but for most liquids the Prandtl numbers are much greater than unity—values of up to 700 are possible.**
- **Therefore the Reynolds analogy is not appropriate for liquids.**
- **Also doesn't account for the varying intensity of molecular and turbulent diffusion in the boundary layer.**

(ii) Prandtl-Taylor Analogy

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- **In independent works, Prandtl [9] and Taylor [10] modified the Reynolds analogy by dividing the boundary layer into two regions**
  - **a viscous sublayer where molecular effects dominate:**

$$\nu \gg \varepsilon_M \text{ and } \alpha \gg \varepsilon_H$$

  - **a turbulent outer layer, where turbulent effects dominate:**

$$\nu \ll \varepsilon_M \text{ and } \alpha \ll \varepsilon_H$$

  - **Notice that neither of these conditions restricts us to *Pr* = 1 fluids.**

Analogy for the Viscous Layer

- **Define the viscous sublayer from $y = 0$ to $y = y_1$**
- **The boundary conditions for this region are**

$$\bar{u}(0) = 0,\ \bar{T}(0) = T_s$$
$$\bar{u}(y_1) = \bar{u}_1,\ \bar{T}(y_1) = \bar{T}_1$$

- **Define the following normalized variables:**

$$U = \frac{\bar{u}}{\bar{u}_1},\ V = \frac{\bar{v}}{\bar{u}_1},\ \theta = \frac{\bar{T} - T_s}{\bar{T}_1 - T_s},\ X = \frac{x}{y_1} \text{ and } Y = \frac{y}{y_1}$$

- **Then, for the viscous sublayer, the ratio of the apparent heat flux and apparent shear stress (Eqn. 8.86) leads to the following:**

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$$T_s - \bar{T}_1 = \frac{q_o''}{\tau_o c_p} Pr \bar{u}_1 \qquad (8.93)$$

Where we have again noted that $q''_{app} / \tau_{app} = q_o'' / \tau_o$ = constant

Analogy for the Outer Layer

- Closely resembles the Reynolds analogy, with $\varepsilon_M = \varepsilon_H$ but this time we assume that the turbulent effects outweigh the molecular effects, equation (8.92).
- The boundary conditions for this region are

$$\bar{u}(y_1) = \bar{u}_1,\ \bar{T}(y_1) = \bar{T}_1$$
$$\bar{u}(y \to \infty) = V_\infty,\ \bar{T}(y \to \infty) = T_\infty$$

- The following normalized variables will make the analogy valid in this region:

$$U = \frac{\bar{u} - \bar{u}_1}{V_\infty - \bar{u}_1},\ V = \frac{\bar{v} - \bar{u}_1}{V_\infty - \bar{u}_1},\ \theta = \frac{\bar{T} - \bar{T}_1}{\bar{T}_\infty - \bar{T}_1},\ X = \frac{x}{L} \text{ and } Y = \frac{y}{L}$$

- Then, for the outer region, the ratio of the apparent heat flux and apparent shear stress (equation 8.86) leads to

$$\bar{T}_1 - T_\infty = \frac{q_o''}{\tau_o c_p}\left(V_\infty - \bar{u}_1\right) \qquad (8.94)$$

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As before, the ratio q''_{app} / τ_{app} is constant, so we have chosen the value at $y = y_1$ (which, as we found for the viscous sublayer, can be represented by q_o'' / τ_o).

- Adding (8.93) and (8.94) gives

$$T_s - T_\infty = \frac{q_o''}{\tau_o c_p} V_\infty \left[\frac{\bar{u}_1}{V_\infty}\left(Pr - 1\right) + 1\right]$$

- Substituting $\tau_o = \frac{1}{2} C_f \rho V_\infty^2$ into the above yields

$$St = \frac{q_o''}{\rho V_\infty c_p \left(T_s - T_\infty\right)} = \frac{C_f / 2}{\frac{\bar{u}_1}{V_\infty}\left(Pr - 1\right) + 1}$$

- The velocity at the edge of the viscous sublayer, $\bar{u}_1$, is still unknown
- Estimate $\bar{u}_1$ using the universal velocity profile. Approximate the edge of the viscous region $u^+ = y^+ \approx 5$
- Then, from the definition of u^+

$$u^+ = 5 = \frac{\bar{u}_1}{V_\infty}\sqrt{\frac{2}{C_f}}$$

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$$\frac{\bar{u}_1}{V_\infty} = 5\sqrt{\frac{C_f}{2}} \qquad (8.95)$$

- Thus, the *Prandtl-Taylor analogy* is

$$St_x = \frac{Nu_x}{Re_x Pr} = \frac{C_f / 2}{\left[5\sqrt{\frac{C_f}{2}}\left(Pr - 1\right) + 1\right]} \qquad (8.96)$$

(iii) von Kármán Analogy

- Theodore von Kármán extended the Reynolds analogy even further to include a third layer – a buffer layer – between the viscous sublayer and outer layer. The result, developed in Appendix D, is

$$St_x = \frac{Nu_x}{Re_x Pr} = \frac{C_f / 2}{1 + 5\sqrt{\frac{C_f}{2}}\left\{\left(Pr - 1\right) + \ln\left[\frac{5Pr + 1}{6}\right]\right\}} \qquad (8.97)$$

(iv) Colburn Analogy

- Colburn [24] proposed a purely empirical modification to the Reynolds analogy that accounts for fluids with varying Prandtl number.

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- He proposed the following correlation through an empirical fit of available experimental data:

$$St_x Pr^{2/3} = \frac{C_f}{2} \qquad (8.98)$$

- The exponent (2/3) on the Prandtl number is entirely empirical, and does not contain any theoretical basis.
- The *Colburn analogy* is considered to yield acceptable results for $Re_x < 10^7$ (including the laminar flow regime) and Prandtl number ranging from about 0.5 to 60.

Example 8.3: Average Nusselt Number on a Flat Plate

Determine the average Nusselt number for heat transfer along a flat plate of length L with constant surface temperature. Use White's model (8.75) for turbulent friction factor, and assume a laminar region exists along the initial portion of the plate.

(1) Observations. This is a mixed-flow type problem, with the initial portion of the plate experiencing laminar flow.

(2) Problem Definition. Determine an expression for the average Nusselt number for a flat plate of length *L*.

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(3) Solution Plan. Start with an expression for average heat transfer coefficient, equation (2.50), and split the integral up between laminar and turbulent regions.

(4) Plan Execution.

(i) Assumptions. (1) Boundary layer assumptions apply, (2) mixed (laminar and turbulent) flow, (3) constant properties, (4) incompressible flow, (5) impermeable flat plate, (6) uniform surface temperature. (7) transition occurs at $x_c = 5*10^5$.

(ii) Analysis.

- The average heat transfer coefficient is found from:

$$\bar{h}_L = \frac{1}{L}\int_0^L h(x)dx \qquad (2.50)$$

- Splits this into laminar and turbulent regions:

$$\bar{h}_L = \frac{1}{L}\left[\int_0^{x_c} h_{lam}(x)dx + \int_{x_c}^{L} h_{turb}(x)dx\right] \qquad (8.99)$$

- From the definition of Nusselt number, we can write the above as:

$$\overline{Nu}_L = \frac{\bar{h}_L L}{k} = \int_0^{x_c} \frac{1}{x} Nu_{x,lam} dx + \int_{x_c}^{L} \frac{1}{x} Nu_{x,turb} dx \qquad (a)$$

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- To find expressions for local Nusselt number, we will use the friction factors for laminar flow, and White's model for turbulent flow (8.75), and apply them to Colburn's analogy (8.98).
- The results are

$$Nu_{x,lam} = 0.332 Pr^{1/3} Re_x^{1/2} \qquad (b)$$

$$Nu_{x,turb} = 0.0135 Pr^{1/3} Re_x^{6/7} \qquad (c)$$

- Substituting these expressions into (a) gives:

$$\overline{Nu}_L = \frac{\bar{h}_L L}{k} = \int_0^{x_c} (0.332) Pr^{1/3} \left(\frac{V_\infty}{\nu}\right)^{1/2} \frac{dx}{\sqrt{x}} + \int_{x_c}^{L} (0.0135) Pr^{1/3} \left(\frac{V_\infty}{\nu}\right)^{6/7} \frac{dx}{x^{1/7}}$$

Which yields

$$\overline{Nu}_L = 0.664 Pr^{1/3} Re_{x_c}^{1/2} + \frac{7}{6}(0.0135) Pr^{1/3}\left(Re_L^{6/7} - Re_{x_c}^{6/7}\right) \qquad (d)$$

- Finally, since $Re_{xc} = 5*10^5$, (d) reduces to:

$$\overline{Nu}_L = \left(0.0158 Re_L^{6/7} - 739\right) Pr^{1/3} \qquad (8.100)$$

(iii) Checking. The resulting Nusselt number correlation is dimensionless.

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(5) Checking. If the laminar length had been neglected, the resulting correlation would be

$$\overline{Nu}_L = 0.0158 Re_L^{6/7} Pr^{1/3} \quad (8.101)$$

This result also makes sense when examining the mixed-flow correlation (8.100). If the plate is very long, such that the majority of the plate is in turbulent flow, the second term in the parentheses becomes negligible, leading to (8.101).

8.5.2 Validity of Analogies

- Generally valid for slender bodies, where pressure gradient does not vary greatly from zero.
- Approximately valid for internal flows in circular pipes as well, although other analogies have been developed specifically for internal flow.
- Although they are derived assuming constant wall temperature, the above correlations work reasonably well even for constant heat flux.
- To address property variation with temperature, evaluate properties at the film temperature:

$$T_f = \frac{T_s + T_\infty}{2} \quad (8.102)$$

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Effect of the Turbulent Prandtl Number

- Analogies also that $Pr_t = 1$. Valid?
- Pr_t as high as 3 near wall, but 0.7–1 outside the viscous sublayer
- Pr_t seems to be affected slightly by pressure gradient, though largely unaffected by surface roughness or the presence of boundary layer suction or blowing.
- A value of $Pr_t \approx 0.85$ is considered reasonable for most flows. This suggests that the analogies should be approximately valid for real flows.

Validity of the Colburn Analogy

- Arguably, the most popular analogy is that of Colburn.
- The analogy is as primitive as the Reynolds analogy, adds no new theoretical insight, and is in fact merely a curve-fit of experimental data.
- Why has this method maintained its usefulness over the decades?
- Easy to use
- More advanced models are based on theoretical assumptions that are, at best, approximations.

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  - o Prandtl-Taylor and von Kármán analogies assume that the viscous sublayer and conduction sublayer are the same thickness.
- Colburn analogy backed by experimental data over a range of conditions and fluids.
- Empiricism is sometimes better than pure theoretical arguments; the test is the experimental data.
- Colburn analogy does have critics. Churchill and Zajic [29] demonstrated that that the Colburn analogy under-predicts the Nusselt number by 30-40% for fluids with Prandtl numbers greater than 7.
- Despite their shortcomings, analogies are fairly straightforward, and facilitate the development of empirical correlations that are often reasonably accurate and easy to use.
- Numerical solutions, on the other hand, are still difficult to obtain and are limited in applicability.
- For these reasons, heat and mass transfer analogies remain in widespread use, and new correlations are still being developed often based on this technique.

8.5.3 Universal Turbulent Temperature Profile

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(i) Near-Wall Profile

- Begin with the turbulent energy equation, (8.39).
- Akin to the Couette flow assumption, we assume that, near the wall, the velocity component $\bar{v}$ ~ 0, as is the temperature gradient $\partial \bar{T} / \partial x$. Thus the left-hand-side of (8.39) approaches zero. Then,

Near the Wall: $$\frac{\partial}{\partial y}\left[\left(\alpha+\varepsilon_H\right)\frac{\partial \bar{T}}{\partial y}\right] \approx 0$$

- This implies that the apparent heat flux is approximately constant with respect to y,

$$\frac{q''_{app}}{\rho c_p} = -\left(\alpha+\varepsilon_H\right)\frac{\partial \bar{T}}{\partial y} \approx \text{constant} \qquad (8.103)$$

- The idea here is the same as we developed for the universal velocity profile: we can solve the above relation for the temperature profile.
- First, recognize that, since $q''_{app} / \rho c_p$ is constant throughout this region, we can replace q''_{app} with q''_o. Then, substituting wall coordinates u^+ and y^+, (8.103) can be rearranged to

$$-\frac{\partial \bar{T}}{\partial y^+}\frac{\rho c_p u^*}{q''_o} = \frac{\nu}{\alpha+\varepsilon_H} \qquad (8.104)$$

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- Define a a temperature wall coordinate as,

$$T^+ \equiv \left(T_s - \bar{T}\right)\frac{\rho c_p u^*}{q''_o} \qquad (8.105)$$

- Then (8.104) becomes,

$$\frac{\partial T^+}{\partial y^+} = \frac{\nu}{\alpha+\varepsilon_H} \qquad (8.106)$$

- We can now integrate the above expression:

$$T^+ = \int_0^{y^+}\frac{\nu dy^+}{\alpha+\varepsilon_H} \qquad (8.107)$$

- We will divide the boundary layer into two regions in order to evaluate this expression.

(ii) Conduction Sublayer

- Very close to the wall, we expect molecular effects to dominate the heat transfer; that is, $\alpha \gg \varepsilon_H$
- Invoking this approximation, (8.107) reduces to:

$$T^+ = Prdy^+ = Pry^+ + C$$

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- The constant of integration, C, can be found by applying the boundary condition that $T^+(y^+=0)=0$. This condition yields $C=0$, so the temperature profile in the conduction sublayer is

$$T^+ = Pry^+, \quad (y^+ < y_1^+) \qquad (8.108)$$

- In the above, y_1^+ is the dividing point between the conduction and outer layers.

(iii) Fully Turbulent Region

- Outside the conduction-dominated region close to the wall, we expect that turbulent effects dominate: $\alpha \ll \varepsilon_H$

$$T^+ = T_1^+ + \int_{y_1^+}^{y^+}\frac{\nu}{\varepsilon_H}dy^+ \qquad (8.109)$$

- Rather than develop some new model for ε_H, we invoke the turbulent Prandtl number:

$$\varepsilon_H = \frac{Pr_t}{\varepsilon_M}$$

- We already have a model for ε_M, and will assume a constant value for Pr_t
- Prandtl's mixing length theory was

$$\varepsilon_M = \kappa^2 y^2 \left|\frac{\partial \bar{u}}{\partial y}\right| \qquad (8.44)$$

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- In terms of wall coordinates, we can write (8.44) as

$$\varepsilon_M = \kappa^2 \left(y^+\right)^2 \nu \frac{\partial u^+}{\partial y^+} \qquad (8.110)$$

- The partial derivative $\partial u^+ / \partial y^+$ can be found from the Law of the Wall, Equation (8.58). Substituting the above and (8.58) into (8.109), we obtain

$$T^+ = \int_{y_1^+}^{y^+} \frac{Pr}{\kappa y^+} dy^+ \qquad (8.111)$$

- Finally, if we assume Pr_t and κ are constants, then (8.110) becomes:

$$T^+ = \frac{Pr_t}{\kappa} \ln\left(\frac{y^+}{y_1^+}\right), \qquad (y^+ > y_1^+) \qquad (8.112)$$

- Kays et al. [30] assumed $Pr_t = 0.85$ and $\kappa = 0.41$, but found that the thickness of the conduction sublayer (y_1^+) varies by fluid.
- White [14] reports a correlation that can be used for any fluid with $Pr \geq 0.7$:

$$T^+ = \frac{Pr_t}{\kappa} \ln y^+ + 13 Pr^{2/3} - 7 \qquad (8.113)$$

- Pr_t is assumed to be approximately 0.9 or 1.0.

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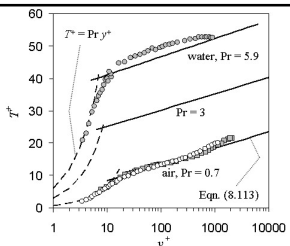

(iv) A 1/7th Law for Temperature

- As with the velocity profile, a simpler 1/7th power law relation is sometimes used for the temperature profile:

$$\frac{\bar{T} - T_s}{T_\infty - T_s} = \left(\frac{y}{\delta}\right)^{1/7} \qquad (8.114)$$

8.5.4 Algebraic Method for Heat Transfer Coefficient

- The existence of a universal temperature and velocity profile makes for a fairly simple, algebraic method to estimate the heat transfer.

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- Begin with the definition of the Nusselt number, which can be expressed using Newton's law of cooling as:

$$Nu_x \equiv \frac{hx}{k} = \frac{q_o'' x}{k\left(T_s - T_\infty\right)} \qquad (8.115)$$

- Invoke the universal temperature profile, T^+: Using the definition of T^+, equation 8.105, we can define the free stream temperature as follows,

$$T_\infty^+ = \left(T_s - T_\infty\right)\frac{\rho c_p u^*}{q_o''} = \left(T_s - T_\infty\right)\frac{\rho c_p V_\infty \sqrt{C_f/2}}{q_o''} \qquad (8.116)$$

- Substituting this expression into (8.115) for $(T_s - T_\infty)$ and rearranging,

$$Nu_x = \frac{\rho c_p V_\infty \sqrt{C_f/2}\, x}{k T_\infty^+}$$

- Then, multiplying the numerator and denominator by ν,

$$Nu_x = \frac{Re_x Pr \sqrt{C_f/2}}{T_\infty^+} \qquad (8.117)$$

- We can now use the universal temperature profile, Eqn. (8.113), to evaluate T_∞^+,

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$$T_\infty^+ = \frac{Pr_t}{\kappa}\ln y_\infty^+ + 13Pr^{2/3} - 7 \qquad (8.118)$$

- A precise value for y_∞^+ is not easy to determine. However, we can make a clever substitution using the Law of the Wall velocity profile, Eqn. (8.58).
- In the free stream, we can evaluate (8.58) as

$$u_\infty^+ = \frac{1}{\kappa}\ln y_\infty^+ + B \qquad (8.119)$$

- Substituting (8.119) into (8.118) for $\ln y_\infty^+$, the Nusselt number relation then becomes

$$Nu_x = \frac{Re_x Pr\sqrt{C_f/2}}{Pr_t\left(u_\infty^+ - B\right) + 13Pr^{2/3} - 7}$$

- We can simplify this expression further. Using the definition of Stanton number, $St_x = Nu_x/\left(Re_x Pr\right)$, selecting $B = 5.0$ and $Pr_t = 0.9$, and noting that the definition of u^+ leads to $u_\infty^+ = \sqrt{2/C_f}$, we can rearrange the relation above to arrive at the final result:

$$St_x = \frac{C_f/2}{0.9 + 13\left(Pr^{2/3} - 0.88\right)\sqrt{C_f/2}} \qquad (8.120)$$

- Note the similarity to the more advanced momentum-heat transfer analogies of Prandtl and Taylor (8.96) and von Kármán (8.97).

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8.5.5 Integral Methods for Heat Transfer Coefficient

- One use for the universal temperature profile is to model the heat transfer using the integral energy equation.
- Consider turbulent flow over a flat plate, where a portion of the leading surface is unheated

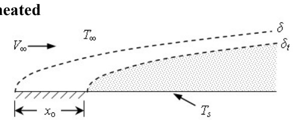

Fig. 8.19

- Can assume a 1/7th power law profile for both the velocity and temperature (equations 8.65 and 8.114), and substitute them into the Energy Integral Equation
- Even with the simplest of assumed profiles, the development is mathematically cumbersome.
- A detailed development appears in Appendix E; the result of the analysis is:

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$$St_x = \frac{Nu_x}{Re_x Pr} = \frac{C_f}{2}\left[1 - \left(\frac{x_o}{x}\right)^{9/10}\right]^{1/9} \qquad (8.121)$$

- Applies to turbulent flow over a flat plate with unheated starting length x_o.
- Note that (8.121) reduces to the Reynolds analogy when $x_o = 0$. This is because the Prandtl number was assumed to be 1 as part of the derivation.
- The model has been used to approximate heat transfer for other fluids as follows. Equation (8.121) can be expressed as

$$Nu_x = \frac{Nu_{x_o=0}}{\left[1 - \left(x_o/x\right)^{9/10}\right]^{1/9}} \qquad (8.122)$$

- In this form, other models for heat transfer, like von Kármán's analogy, could be used to approximate $Nu_{x_o=0}$ for $Pr \neq 1$ fluids.

8.5.6 Effect of Surface Roughness on Heat Transfer

- We would expect roughness to increase the heat transfer, like it did for the friction factor.
- However, the mechanisms for momentum and heat transfer are different.

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- As roughness increases, the viscous sublayer diminishes, to such an extent that for a fully rough surface the viscous sublayer disappears altogether.
  - The turbulent fluid elements are exchanging momentum with surface directly (like *profile* or *pressure drag*), and the role of molecular diffusion (i.e., *skin friction*) is diminished.
- Heat transfer, on the other hand, relies on molecular conduction at the surface, no matter how rough the surface, or how turbulent the flow.
  - There is no "pressure drag" equivalent in heat transfer.
  - Moreover, fluid in the spaces between roughness elements is largely stagnant, and transfers heat entirely by molecular conduction.
  - The conduction sublayer, then, can be viewed as the average height of the roughness elements.
  - The stagnant regions between roughness elements effectively create a resistance to heat transfer, and is the major source of resistance to heat transfer [27].
- Bottom line: we can not expect roughness to improve heat transfer as much as it increases friction.

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- This is also means that we can not predict the heat transfer by simply using a friction factor for rough plates along with one of the momentum-heat transfer analogies.

What Influences Heat Transfer on a Rough Plate?

- The roughness size k
  - Expect that roughness size has no influence until it extends beyond the viscous and conduction sublayers.
  - Its influence reaches a maximum beyond some roughness size (the *fully rough* limit).
- The Prandtl number
  - Since molecular conduction is important.
  - Fluids with higher Prandtl number (lower conductivity) would be affected more by roughness. Why? The lower-conductivity fluid trapped between the roughness elements will have a higher resistance to heat transfer. Also, the conduction sublayer is shorter for these fluids, so roughness elements penetrate relatively further into the thermal boundary layer.
  - In contrast, for a liquid metal, the conduction sublayer may fully engulf the roughness elements, virtually eliminating their influence on the heat transfer.

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- Kays et al. [30] develop a correlation for rough plate, which is equivalent to

$$St = \frac{C_f}{2}\left[Pr_t + C\left(k_s^+\right)^{0.2} Pr^{0.44}\sqrt{C_f/2}\right]^{-1} \qquad (8.123)$$

- where $k_s^+ = k_s u^* / \nu$ is based on the equivalent sand-grain roughness k_s and C is a constant that depends on roughness geometry.
- Bogard et al. [31] showed that this model compared well with experimental data from roughened turbine blades.
  - Showed a 50% increase in heat transfer over smooth plates.
  - Demonstrated that increasing roughness beyond some value showed little increase in the heat transfer.

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